Scheme of work

Cambridge IGCSE®
Additional Mathematics (US)
0459

Cambridge IGCSE

Cambridge Secondary 2

CAMBRIDGE
International Examinations

Contents

Scheme of work – Cambridge IGCSE® Additional Mathematics (US) 0459

Overview

This scheme of work provides ideas about how to construct and deliver a course. The 2013 syllabus for 0459 has been broken down into teaching units with suggested teaching activities and learning resources to use in the classroom.

Recommended prior knowledge
It is recommended that learners are familiar with the content of the 0444 extended syllabus or have followed similar courses.

Outline
The units within scheme of work are:

Unit 1: Number
Unit 2: Algebra
Unit 3: Functions
Unit 4 Geometry
Unit 5: Transformations and vectors
Unit 6: Coordinate geometry
Unit 7: Trigonometry
Unit 8: Probability
Unit 9: Statistics

Teaching order
Units have been mapped to the relevant syllabus sections. However, this will not provide a natural teaching order. The following teaching order of units is recommended, although other permutations are entirely possible.

Unit 2: Algebra
Unit 3: Functions
Unit 5: Transformations and vectors
Unit 7: Trigonometry
Unit 1: Number
Unit 6: Coordinate geometry
Unit 4: Geometry
Unit 8: Probability
Unit 9: Statistics

Common Core State Standards (CCSS)
In each unit the relevant standards are indicated in bold blue lettering **CCSS**, in the first column. This allows teachers to identify how standards are met in particular activities.

Teacher support
Teacher Support is a secure online resource bank and community forum for Cambridge teachers. Syllabus 0459 is a new syllabus and specimen papers are available for teachers on Teacher Support at http://teachers.cie.org.uk. Teachers may want to refer to the Cambridge IGCSE Additional Mathematics (syllabus 0606) and Cambridge IGCSE Mathematics (US) 0444 support materials for additional resources. These syllabuses are referred to throughout the 0459 scheme of work.

An editable version of this scheme of work is available on Teacher Support. Go to http://teachers.cie.org.uk. The scheme of work is in Word doc format and will open in most word processors in most operating systems. If your word processor or operating system cannot open it, you can download Open Office for free at www.openoffice.org

Resources

Textbooks:

Mathematics: Pure Mathematics 1
Author: Bostock, L and Chandler, S
ISBN: 0859500926
Published in 1978.
Published by Nelson Thornes, UK
www.nelsonthornes.com

Geometry Essentials for Dummies
Author: Ryan, M
ISBN: 9781118068755
Published in 2011
Published by Wiley, USA
www.wiley.com

Understanding Pure Mathematics
Authors: Sadler, A. J and Thorning D. W. S
ISBN: 9780199142439
Published in 1987
Published by Oxford University Press, UK
http://ukcatalogue.oup.com/

Websites:
This scheme of work includes website links providing direct access to internet resources. Cambridge International Examinations is not responsible for the accuracy or content of information contained in these sites. The inclusion of a link to an external website should not be understood to be an endorsement of that website or the site's owners (or their products/services).

The particular website pages in the learning resource column were selected when the scheme of work was produced. Other aspects of the sites were not checked and only the particular resources are recommended.

Unit 1
http://betterexplained.com/articles/a-visual-intuitive-guide-to-imaginary-numbers/
www.purplemath.com/modules/complex.htm
http://betterexplained.com/articles/intuitive-arithmetic-with-complex-numbers/
www.geogebra.org/cms/en
www.analyzemath.com/complex/questions.html
http://wiki.geogebra.org/en/Complex_Numbers
www.purplemath.com/modules/complex3.htm
www.mathwarehouse.com/quadratic/quadratic-formula-calculator.php
www.nationalstemcentre.org.uk/elibrary/file/594/NA8.pdf
www.nationalstemcentre.org.uk/elibrary/maths/resource/530/aqa-fp1-matrices-transformations
www.mathplanet.com/education/geometry/transformations/transformation-using-matrices
http://wiki.geogebra.org/en/Matrices

Unit 2
www.khanacademy.org/math/algebra/polynomials/v/dividing-polynomials-with-remainders
www.purplemath.com/modules/polydiv3.htm
www.nationalstemcentre.org.uk/elibrary/file/17916/mc-TY-polydiv-2009-1.pdf
www.purplemath.com/modules/synthdiv.htm
www.purplemath.com/modules/solvpoly.htm
http://tutorial.math.lamar.edu/Classes/Alg/Factoring.aspx
www.nationalstemcentre.org.uk/elibrary/file/6410/A11.pdf
www.geogebratube.org/material/show/id/12541
www.geogebratube.org/student/m10583
www.geogebratube.org/student/m7323
www.tes.co.uk/teaching-resource/Simultaneous-Equations-6146699/

Unit 3
www.nationalstemcentre.org.uk/elibrary/file/17986/mc-TY-introfns-2009-1.pdf
www.nationalstemcentre.org.uk/elibrary/file/17944/mc-TY-inverse-2009-1.pdf
www.nationalstemcentre.org.uk/elibrary/file/17948/mc-TY-trig-2009-1.pdf
www.nationalstemcentre.org.uk/elibrary/file/17986/mc-TY-introfns-2009-1.pdf
www.tes.co.uk/teaching-resource/When-does-fg-equals-gf-Risp-18-6056316/
www.khanacademy.org/math/algebra/algebra-functions/v/function-inverses-example-2
www.purplemath.com/modules/invrsfcn7.htm
www.khanacademy.org/math/algebra/algebra-functions/v/function-inverses-example-3
www.nationalstemcentre.org.uk/elibrary/file/17944/mc-TY-inverse-2009-1.pdf
www.geogebratube.org/material/show/id/5538
www.purplemath.com/modules/graphabs.htm
www.khanacademy.org/math/algebra/algebra-functions/v/recognizing-odd-and-even-functions
www.khanacademy.org/math/algebra/algebra-functions/v/recognizing-odd-and-even-functions
www.purplemath.com/modules/fcnnot3.htm
www.khanacademy.org/math/algebra/algebra-functions/e/even_and_odd_functions
www.purplemath.com/modules/fcncomp5.htm
www.ugrad.math.ubc.ca/coursedoc/math100/notes/zoo/composite.html
www.geogebratube.org/student/m4976
http://betterexplained.com/articles/an-intuitive-guide-to-exponential-functions-e/
www.nationalstemcentre.org.uk/elibrary/file/17942/mc-TY-explogfns-2009-1.pdf
www.tes.co.uk/ResourceDetail.aspx?storyCode=6033007
www.s253053503.websitehome.co.uk/risps/risp31.html
http://nrich.maths.org/5831
http://nrich.maths.org/334

Unit 4
www.mathopenref.com/congruenttriangles.html
www.mathwarehouse.com/classroom/worksheets/congruent_triangles/Triangle_proof_ASA-SAS.pdf
www.mathwarehouse.com/geometry/congruent_triangles/congruent-parts-CPCTC.php
www.letspracticegeometry.com/wp-content/uploads/2011/11/proofs-involving-CPCTC.pdf
www.glencoe.com/sites/texas/student/mathematics/assets/interactive_lab/geometry/G_04/G_04_dev_100.html
http://nrich.maths.org/6355
http://nrich.maths.org/272
http://nrich.maths.org/700/note

Unit 5
www.tes.co.uk/teaching-resource/Cartesian-components-3004262/
www.tes.co.uk/teaching-resource/Vectors-multiple-choice-6161352/
www.khanacademy.org/science/physics/v/introduction-to-vectors-and-scalars
www.mathwarehouse.com/vectors/
www.geogebratube.org/student/m12566
www.mathwarehouse.com/vectors/
www.teachengineering.org/view_activity.php?url=collection/cub_/activities/cub_navigation/cub_navigation_lesson02_activity1.xml#mats
https://education.staffordshire.gov.uk/NR/.../ResultantVelocity.xls
www.nationalstemcentre.org.uk/dl/cb9e29b55c6c271e9eeac0b5c1bccfe1f993a77a/520-1_Aircraft_Navigation.pdf

Unit 6
www.purplemath.com/modules/slopyint.htm
www.tes.co.uk/ResourceDetail.aspx?storyCode=6111880
www.tes.co.uk/teaching-resource/Equations-of-straight-lines-6148248/
http://demonstrations.wolfram.com/ConicSectionsTheDoubleCone/
www.geogebratube.org/student/m7505
www.geogebratube.org/material/show/id/5020
www.purplemath.com/modules/parabola.htm
www.khanacademy.org/math/algebra/conic-sections/v/conic-sections--intro-to-ellipses
www.purplemath.com/modules/ellipse.htm
www.geogebratube.org/student/m10565
www.purplemath.com/modules/hyperbola.htm
www.geogebratube.org/student/m10566
www.shodor.org/interactivate/activities/ConicFlyer/

Unit 7
www.nationalstemcentre.org.uk/dl/1d50f77a0fbe87381303373da5af07d6094041ba/17969-mc-TY-radians-2009-1.pdf
www.s253053503.websitehome.co.uk/risps/tes-risp-23.pdf
http://nrich.maths.org/681
http://map.mathshell.org/materials/download.php?fileid=1284
www.geogebratube.org/student/m3342
www.haesemathematics.com.au/samples/ibmyp5plus-2_18.pdf
www.geogebratube.org/student/m12795
www.worsleyschool.net/science/files/cast/castdiagram.html
www.purplemath.com/modules/trig.htm
www.geogebratube.org/student/m1395
www.haesemathematics.com.au/samples/ibmyp5plus-2_18.pdf
www.nationalstemcentre.org.uk/dl/de3dceb972208ceb7024ca8bb56259452d8aea76/6411-A12.pdf
www.youtube.com/watch?v=o7Ho1bMWhG8
http://map.mathshell.org/materials/lessons.php?taskid=427&subpage=concept

www.geogebratube.org/student/m14662
www.nctm.org/uploadedFiles/Journals_and_Books/Books/FHSM/RSM-Task/Tidal_Waves.pdf
www.tes.co.uk/ResourceDetail.aspx?storyCode=6056103
www.nationalstemcentre.org.uk/dl/a22042d9764f5223a67d8931a85f2edc6881484b/17967-mc-TY-addnformulae-2009-1.pdf
www.tes.co.uk/ResourceDetail.aspx?storyCode=6030084

Unit 8
www.cancer.org/Cancer/CancerCauses/TobaccoCancer/tobacco-related-cancer-fact-sheet
www.making-statistics-vital.co.uk/
www.tes.co.uk/teaching-resource/Statistics-1-Arrangements-6033012/
http://betterexplained.com/articles/easy-permutations-and-combinations/
www.tes.co.uk/teaching-resource/Permutation-and-Combination-with-the-Lottery-6126510/
www.nationalstemcentre.org.uk/dl/dfb33ea6bb84d775c61123bd468c7338b5aaa119/18560-msv-5.pdf
www.geogebratube.org/student/m4054
http://map.mathshell.org/materials/lessons.php?taskid=438&subpage=problem
http://map.mathshell.org/materials/lessons.php?taskid=409&subpage=problem
www.khanacademy.org/math/probability/v/term-life-insurance-and-death-probability

Unit 9
www.amstat.org/education/gaise/GAISEPreK12_LevelB.pdf
www.amstat.org/education/gaise/GAISEPreK12_LevelC.pdf
https://ccgps.org/S-ID_625Z.htm
www.amstat.org/education/stew/pdfs/ColorsChallenge.docx
www.amstat.org/education/stew/pdfs/WhatDoestheNormalDistributionSoundLike.pdf
www.geogebratube.org/material/show/id/11896
http://stattrek.com/statistics/two-way-table.aspx
http://www4.ncsu.edu/~muse/Teaching/ST305/Lectures/Ch02-pt2.pdf
www.geogebratube.org/student/m14014
http://ccsstoolbox.agilemind.com/animations/standards_content_visualizations_algebra_1.html
www.cpm.org/pdfs/standards/stats/Stats%20Unit%207%20TV.pdf

Unit 1: Number

Recommended prior knowledge
It is recommended that learners study Unit 2 (Algebra) prior to Unit 1 as this will revise and reinforce the skills they will need to apply to working with complex numbers. It is also recommended that learners study Unit 7 (Trigonometry) as this will ensure they have the skills required for understanding the argument of complex number and also the geometrical effect of multiplication on the argument. Also it is desirable that Unit 5 (Transformations and Vectors) be covered prior to the study of complex numbers and matrices as this will enhance learners' understanding of the geometrical effects of operations on complex numbers and the benefit of having done vectors in advance of matrices should be fairly apparent.

Context
The unit has been broken into three subsections, which map to the syllabus as Subsection A (Exponents and Radicals), Subsection B (Complex Number), Subsection C (Matrices). These subsections can be taught separately or together as required. Skills within Subsection A will be necessary for handling syllabus ref. 6.7

Outline
Syllabus ref. 1.9 of this unit builds on skills learners should already have covered in the extended 0444 syllabus, particularly 1.7/1.8 and 2.4. The skills in syllabus 0444 are developed and the presentation should be more challenging as learners take a step up in the level of mathematics they are experi.e.ncing. The remainder of the material in this unit – arithmetic and geometry of complex number and using and applying matrices – is new to learners.

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
Subsection A (Exponents and Radicals)			
1.9 **CCSS:** **N-RN1** **N-RN2**	Perform simple operations with indices and with radicals, including rationalizing the denominator.	**General guidance** *Linking to syllabus 0444_1.7/1.8 and 2.4* *To some extent this is just a coding change i.e. a power 'a half' and 'square root' sign. Convince pupils using examples that can be shown to be true using the index laws and then practice switching between the codes.* All skills covered in syllabus 0444 – the four operations applied to radicals and laws of indices – will be assumed here. We build on the use	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		of exponents with numerical bases, and extend this further, while also including further algebraic skills, including solving more involved radical equations. Rationalizing the denominator is a new skill dictated purely by mathematical convention. It will be useful to revise the difference of two squares to show why, for example $1+\sqrt{3}$ is the multiplier to use in order to rationalize $1-\sqrt{3}$. Make sure that learners appreciate that multiplying both the numerator and the denominator by the "same" number with opposite sign (the square root conjugate) means that the original expression is being multiplied by a strategic form of 1 and therefore identity is maintained. Even though this syllabus reference is under the umbrella of "Number", learners will be expected to use index laws to simplify ***algebraic*** expressions, which may be rational. They will need to be able to work seamlessly between exponent and radical forms, deciding on which form is appropriate to best solve the problem at hand, which may include solving equations involving radicals and indices. They will also need to appreciate that $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$ and vice versa and also $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$ and vice versa, algebraically as well as numerically. It is very important, therefore, a really good grounding in working with exponent and radical forms with numerical and simple algebraic bases is achieved/revised before moving on to more involved forms. **Teaching activities** Investigation is a good approach here to discover what rationalizing the denominator is all about and why we do it, as well as manipulating expressions with radicals. Start with questions like, for example, Simplify each of the following and hence show that all three expressions	Specimen Paper 1 Question 5 Specimen Paper 2 Question 3

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		are equal: (i) $\frac{\sqrt{32}}{4}$ (ii) $4\sqrt{2}-\sqrt{18}$ (iii) $\frac{\sqrt{10}}{\sqrt{5}}$ Parts (i) and (ii) should be nothing more than revision of the skills previously acquired. The last part has two natural approaches using either of $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ or $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$. This could then be used to extend into rationalizing the denominator. Start rationalizing with single surd denominators e.g. Write $\frac{3}{\sqrt{5}}$ with a rational denominator. Then extend to binomial terms in the numerator e.g. $\frac{3+\sqrt{5}}{\sqrt{5}}$ and then to binomial terms in both the numerator and denominator e.g. $\frac{3+\sqrt{5}}{1+\sqrt{5}}$. Once these basic skills have been acquired, learners can be moved on to move involved expressions such as $\frac{(3+\sqrt{5})^2}{1+\sqrt{5}}$. Once mastery has been achieved in numerical skills apply laws of exponents and radicals to algebraic expressions and equations. Start with simpler skills e.g. Solve $\sqrt[3]{3x+5}=2$. This could be solved by inspection but formalize the method by **Step1** Raise both sides to the index of the radical **Step2** Solve the resulting equation **Step3** Check the answer is valid to avoid extraneous solutions Solve $4+\sqrt{x-4}=5$ **Step1** Isolate the radical $\sqrt{x-4}=1$ **Step2** Raise both sides to the index of the radical $x-4=1^2$	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Step3 Solve the resulting equation $x = 5$ **Step4** Check $4+\sqrt{5-4}=5$ ✓ Solve $\sqrt{x}+\sqrt{1-x}=1$ **Step1** Isolate a radical $\sqrt{1-x}=1-\sqrt{x}$ **Step2** Raise both sides to the index of the isolated radical $1-x=(1-\sqrt{x})^2 \quad 1-x=1-2\sqrt{x}+x$ **Step3** Isolate the radical $x=\sqrt{x}$ **Step4** Raise both sides to the index of the isolated radical $x^2=x$ **Step5** Solve $x^2-x=0 \quad x(x-1)=0 \quad x=0$ or $x=1$ **Step6** Check validity $\sqrt{0}+\sqrt{1-0}=\sqrt{1}=1$ ✓ $\quad \sqrt{1}+\sqrt{1-1}=\sqrt{1}=1$ ✓ The validity of the solutions can be checked algebraically or graphically on a calculator. Once these skills have been practiced, move on to expressions/equations such as those in the specimen paper that mix exponents and radicals with learners choosing an appropriate route to solve the problem given e.g. Simplify $\dfrac{\sqrt{4x-3}+(4x-3)^{-\frac{3}{2}}}{\sqrt{4x-3}}$ to solve $\dfrac{\sqrt{4x-3}+(4x-3)^{-\frac{3}{2}}}{\sqrt{4x-3}}=\dfrac{5}{4}$	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
Subsection B (Complex Number)			
1.1 **CCSS:** **N-CN1**	Understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, modulus, argument, conjugate, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal.	**Notes and exemplars** Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + ib$ with a and b real. **General guidance** The easiest way to start this topic is to begin by looking at the solution to a simple equation such as $x^2+1 = 0$. To consider that this equation has no solution in any of the number systems studied to date. Discuss that all numbers currently worked with (i.e. rationals and irrationals) belong to the Real number system. From this, define $i^2 = -1$ and hence the imaginary number, $i = \sqrt{-1}$. The *betterexplained* webpages are an excellent resource for intuitive understanding of imaginary numbers. Then formally define $\sqrt{-n}$. Be careful to point out that when dealing with $\sqrt{-n}$, you **must** split it into $\sqrt{-1} \times \sqrt{n} = i\sqrt{n}$, i.e. deal with the i first and that, for example, $i^2 = \left(\sqrt{-1}\right)^2 = \sqrt{(-1)^2} = \sqrt{1} = 1$ is a false deduction in the realm of complex numbers. From that the set of imaginary numbers can be considered. Start with operations on imaginary numbers and link this to algebraic skills. Allow learners to develop the idea that some operations on imaginary numbers can produce real results. Once imaginary numbers have been introduced, the system of complex numbers can then be defined. Discussion can then be undertaken to establish both the set of reals as a subset of the complex numbers and similarly the set of imaginaries.	A visual, intuitive guide to imaginary numbers: http://betterexplained.com/articles/a-visual-intuitive-guide-to-imaginary-numbers/

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Teaching activities As an investigation consider powers of i and investigate their cyclical nature. Once a repeating pattern of results has become apparent, consider how to find, for example, the simplified form of i^{97}, by dividing 97 by 4 and considering the remainder as the appropriate power. *Purplemath* provides some good examples of this. Once complex numbers of the form $a + \mathrm{i}b$ have been introduced, learners can undertake sorting numbers into sets (nested Venn diagrams would be good here) – which are entirely real, which are entirely imaginary, which are complex.This could be a timed or competitive exercise in groups or a whole group activity, but it is an important concept and time spent at this stage will be rewarded later. PowerPoint presentations could be well utilized here, generating various purely real, imaginary or complex numbers to be grouped into correct sets so that learners could work alone or in groups as appropriate to the task set.	Complex numbers introduction: www.purplemath.com/modules/complex.htm
		General guidance Modulus, argument will be defined in subsequent sections. The conjugate can be defined as soon as the concept of a complex number has been introduced and grasped. It is a basic concept, but of such importance that it should be introduced prior to arithmetic being tackled. A simple statement that $z = a + \mathrm{i}b$ and $\bar{z} = a - \mathrm{i}b$ are (complex) conjugates here will suffice, as the usefulness of and strength of relationship between the conjugate pairs will become apparent through the work in subsequent syllabus references.	Some text books use z* for complex conjugates.

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		General guidance Equal complex numbers can be introduced at this stage (and linked to equal vectors, if vectors have previously been covered) Also reinforce here that each and every complex number has a real part (Re) and an Imaginary part (Im). **Teaching activities** Set up and solve simple identities such as 3 + 4i = *a* + i*b*, find the value of *a* and of *b*. Sorting activities will also reinforce the structure of complex numbers as having a real part and an imaginary part. These activities would ideally be pair or group based, but could be individual through use of technology if individual teaching was the goal here.	
1.2 **CCSS: N-CN2**	Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, multiply, and divide two complex numbers expressed in the form $x + iy$.	**General guidance** It is helpful for learners to have revised or been reminded of handling arithmetic of irrational numbers and also algebraic manipulation such as collecting like terms (Re parts are "like", similarly Im parts) and FOIL (multiply First terms, Outside terms, Inside terms, Last terms) before tackling this topic as the skills they need are similar. Real numbers can be ordered in terms of size. Complex numbers are 2 dimensional and therefore cannot be ordered on a number line – they require a complex plane. Using an Argand diagram, the size of a complex number can be defined in terms of its distance from an origin – i.e. the modulus – and this found by applying Pythagoras' theorem. The *betterexplained* webpages give good insight into arithmetic of complex numbers in this 2-dimensional sense. The conjugate can be plotted on the Argand diagram and the transformation of the original number as a reflection in the Real axis should become apparent.	*Pure Mathematics 1* Chapter 14 Intuitive arithmetic with complex numbers: http://betterexplained.com/articles/intuitive-arithmetic-with-complex-numbers/

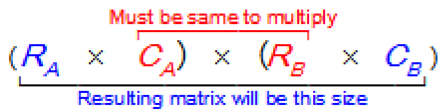
Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
1.3 **CCSS: N-CN3**	Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.	The conjugate can be used to find the modulus as follows, using the definitions already given $z(\bar{z}) = (a + ib)(a - ib) = a^2 + b^2 = \|z\|^2$ The conjugate can also be used to perform division in a very similar way to rationalising the denominator of a fraction with an irrational denominator. i.e. multiply the numerator and denominator by the complex conjugate of the denominator. The argument needs to be introduced next in order to convert to modulus-argument (polar) form. This form will be a natural extension of ideas covered in Units 5 and 7 (Vectors and Trig). Regardless, the concept is intuitively straightforward with applications of trigonometry to acute angles and can be extended to the principal arg, between $-\pi$ and π radians.	Specimen Paper 2 Question 10
1.4 **CCSS: N-CN4**	Represent complex numbers geometrically in the complex plane in rectangular and polar form, and convert between the rectangular and polar forms of a complex number.	The need to draw a quadrant diagram to ensure that the correct argument is used for the complex number given should be emphasized firmly! **Teaching activities** Learners need lots and lots of exploration in the world of complex numbers, just as young children do with real arithmetic. Give lots of investigative work to start with, but also lots of practice with pen and paper exercises to reinforce the concepts established. The learning resources detailed at the *analyzemath* website will be useful for these purposes.	Questions on complex numbers: www.analyzemath.com/complex/questions.html
1.5 **CCSS: N-CN5**	Understand in simple terms the geometrical effects of conjugating a complex number and of adding, subtracting, and multiplying two complex numbers, and use properties of this representation.	**Notes and exemplars** e.g., $(-1 + \sqrt{3}i)^3 = 8$ because $(-1 + \sqrt{3}i)$ has modulus 2 and argument 120° **General guidance** Considering the geometrical transforms as follows: • Conjugating as **reflection** in the real axis	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		• Addition and subtraction (as additive inverse) as per vector addition and subtraction as **translations** • Multiplication in terms of **rotation**. Show by example and generalize from that, that when two complex numbers are multiplied their arguments are added and their moduli multiplied. It will be helpful to have covered the tan (A + B) here to establish the effect multiplication by a complex number with an imaginary part has on another complex number with an imaginary part with a group whose skills are of a good level. The notes and examples (above) can be then established by finding the modulus and argument of the base complex number $-1 + i\sqrt{3}$ and showing the effect squaring and then cubing has upon the modulus and argument. **Teaching activities** Set up investigations Addition and subtraction should be intuitive or certainly not new in 2 dimensional forms at this stage. Multiplication: • first an imaginary by a real and consider the 90 degree rotation anti-clockwise of the number in the complex plane • complex by a real and consider the effect on modulus and argument • then a complex by an imaginary (i first then others) and consider the effect on modulus and argument • then a complex by a complex with both Re and Im parts. The use of technology to reinforce these concepts will be vital and it is suggested that graphical calculators or appropriate software packages are used in order to do that (*Geogebra* is useful for this and can show rectangular and polar forms).	Geogebra – for help with inputting complex values see http://wiki.geogebra.org/en/Complex_Numbers

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
1.6 **CCSS: N-CN6**	Calculate the distance between numbers represented in the complex plane and the midpoint of a line segment.	**General guidance** In rectangular form this simplifies again in essence to an application of Pythagoras' Theorem. Once the complex number has been plotted on the Argand diagram the problem resolves into one of co-ordinate geometry and this may be a good area to move into next if not already covered.	
1.7 **CCSS: N-CN7**	Solve quadratic equations with real coefficients that have complex solutions.	**General guidance** This brings in excellent opportunities to draw in known skills in the area of solving quadratic equations – all those equations that were previously "not solvable" suddenly have solutions. *Purplemath* gives a good introduction to this, as does the National Stem Center activity – Solving quadratic equations. **Teaching activities** Using graphical calculators or software (see *mathwarehouse*) to support learning, learners can teamwork and brainstorm solving quadratic equations, manually and using technology, and present their results to the class as a whole. This activity should enable learners to become familiar and comfortable with such a novel concept.	Complex numbers & the quadratic formula: www.purplemath.com/modules/complex3.htm Solving quadratic equations: www.nationalstemcentre.org.uk/elibrary/file/594/NA8.pdf Quadratic formula calculator & solver: www.mathwarehouse.com/quadratic/quadratic-formula-calculator.php Specimen Paper 1 Question 3
1.8 **CCSS: N-CN8**	Extend polynomial identities to complex numbers.	**Notes and exemplars** e.g. rewrite $x^2 + 4$ as $(x + 2\text{i})(x - 2\text{i})$ **General guidance** The easiest way to deal with the example above is to rewrite as the difference of 2 squares $x^2 - (-4) = x^2 - (4\text{i}^2) = x^2 - (2\text{i})^2 = (x + 2\text{i})(x - 2\text{i})$ If asked to prove such relationships, learners should take care to work from the left to the right and not work backwards from right to left.	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
Subsection C (Matrices)			
1.10 **CCSS: N-VM6**	Display information in the form of a matrix of any order and interpret the data in a given matrix.	**General guidance** This topic is more intuitively understood if it is introduced via a practical example such as the stocks of items held in a store. Give a list of data and construct a matrix from it such as $\begin{pmatrix} 5 & 6 & 12 & 11 & 0 \\ 0 & 13 & 15 & 6 & 1 \\ 11 & 17 & 14 & 7 & 4 \end{pmatrix}$ Define a matrix is an array or rows and columns and the concept of a matrix of order m by n with m rows and n columns can be immediately considered. Each entry in the matrix representing a specific quantity can be reinforced through this type of example and it can be extended to include addition and subtraction as follows: The store manager wished to ensure that her stock levels are constant at 20 units of each item, write down the matrix representing the items she needs to order to ensure that the stock levels are maintained. The matrix $\begin{pmatrix} 15 & 14 & 8 & 9 & 20 \\ 20 & 7 & 5 & 14 & 19 \\ 9 & 3 & 6 & 13 & 16 \end{pmatrix}$ and the idea of matrix addition can be established via $\begin{pmatrix} 5 & 6 & 12 & 11 & 0 \\ 0 & 13 & 15 & 6 & 1 \\ 11 & 17 & 14 & 7 & 4 \end{pmatrix} + \begin{pmatrix} 15 & 14 & 8 & 9 & 20 \\ 20 & 7 & 5 & 14 & 19 \\ 9 & 3 & 6 & 13 & 16 \end{pmatrix} = \begin{pmatrix} 20 & 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 & 20 \end{pmatrix}$. Subtraction could be introduced via the withdrawals of items made on a particular day and it can be established at this point that the size of the matrices must be the same in order that addition/subtraction be possible.	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
1.11 **CCSS:** **N-VM8** **N-VM9** **N-VM11**	Solve problems involving the calculation of the sum and product (where appropriate) of two matrices and interpret the results.	**Notes and exemplars** Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties. **General guidance** Again, the storekeeper example can be used here to consider matrix multiplication in its simplest form by considering small number of stock items and their costs and using matrix multiplication to work out the total cost. For example, the cost of a PC mouse is \$22, a keyboard is \$16, a USB hub is \$25 and a monitor is \$105. Work out the total cost of 1 PC mouse, 3 keyboards, 5 USB hubs and 1 monitor. The matrix $\begin{pmatrix} 1 & 3 & 5 & 1 \end{pmatrix}$ represents the number of each item required and $\begin{pmatrix} 22 \\ 16 \\ 25 \\ 105 \end{pmatrix}$ the cost of each item. Show that the total cost is found by $\begin{pmatrix} 1 & 3 & 5 & 1 \end{pmatrix}\begin{pmatrix} 22 \\ 16 \\ 25 \\ 105 \end{pmatrix} = \begin{pmatrix} 300 \end{pmatrix}$. Now extend this by adding a new row to the number of items representing the quantities of each of the items purchased by a second customer e.g. $\begin{pmatrix} 1 & 3 & 5 & 1 \\ 0 & 2 & 3 & 0 \end{pmatrix}\begin{pmatrix} 22 \\ 16 \\ 25 \\ 105 \end{pmatrix} = \begin{pmatrix} 300 \\ 107 \end{pmatrix}$ the concept of multiplying and adding elements of the matrices is established. Consider a matrix of item numbers and costs such as	Specimen Paper 2 Question 7 Resources for practical situation are fairly hard to come by, learners could be encouraged to research and come up with some of their own examples which could then be shared with their peers (possible group work).

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		$\begin{pmatrix}1 & 3 & 5 & 1\\0 & 2 & 3 & 0\end{pmatrix}\begin{pmatrix}22\\16\\25\\ \end{pmatrix}$ – this cannot be found since the cost of one of the items is missing. Now establish that to multiply **A** by **B** the number of columns of matrix **A** has to be the same as the number of rows of matrix **B**.  **Teaching activities** Learners need to be presented with lots of practical situations from which to derive sets of matrices in order to solve problems.	
1.12 **CCSS: N-VM7**	Calculate the product of a scalar quantity and a matrix.	**General guidance** The idea of the stock in a store can be continued here to demonstrate the effect of multiplying by a scalar by, for example, the store keeper wishing to double her order of stock in a month to cover the holiday season and show the effect that would have (and there are of course natural links to multiplying a vector by a scalar here, which can be utilized should vectors have already been considered).	
1.13 **CCSS: N-VM10**	Use the algebra of 2 × 2 matrices (including the zero and identity matrix).	**General guidance** Define a 2 by 2 matrix as a square matrix. By verification on 2 by 2 matrices **A**, **B** and **C** for example, learners will need to appreciate that matrices follow • the associative laws **A**+(**B**+ **C**) = (**A**+**B**)+**C** and **A**(**BC**) = (**AB**)**C** • the distributive law **A**(**B**+ **C**) = **AB**+**AC** • the commutative law **A**+**B** = **B**+**A** but that **AB** ≠ **BA** and so matrices are not commutative with respect to	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		multiplication. Learners will then need to grasp that, just as 0 is the additive identity for numbers (i.e. leaves the original value unchanged), the 2 by 2 **zero matrix** $\mathbf{O}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ is the additive identity for 2 by 2 matrices. Also that as 1 is the multiplicative identity for numbers, the 2 by 2 **identity matrix** $\mathbf{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Also, in the usual way, for a square matrix **A**, $\mathbf{AA}=\mathbf{A}^2$, $\mathbf{AAA}=\mathbf{A}^3$ etc.	
1.14 **CCSS:** **N-VM10** **A-REI8** **A-REI9**	Calculate the determinant and inverse of a non-singular 2 × 2 matrix and solve simultaneous linear equations.	**Notes and Exemplars** The determinant of a square matrix is non-zero if, and only if, the matrix has a multiplicative inverse. **General guidance** The Notes and Exemplars here give the mechanism for learners to determine whether a matrix is singular or whether it has an inverse. Learners will already be familiar with the notation for an inverse function and therefore the notation $\mathbf{A}^{-1}$ to represent the inverse of the matrix **A**. **Note:** $\mathbf{A}^{-1}$ is the *multiplicative* inverse but is simply called the inverse, since it is the most useful (the additive inverse being negative **A**). Just as with numbers, the product of a number and its multiplicative inverse is unity. Hence it follows that $\mathbf{AA}^{-1}$ or indeed $\mathbf{A}^{-1}\mathbf{A}$ must equal **I**. In order for this to be achieved, the determinant must be established as $\det\mathbf{A} = ad - bc$ for $\mathbf{A} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$. This can be done by multiplying a matrix and its inverse together and showing that a multiple of the identity matrix is obtained.	Specimen Paper 1 Question 7

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		$\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$ can be used to establish swapping the elements on the leading diagonal and changing the signs of the other two elements. $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -0.5 & 1 \\ 1.5 & -2 \end{pmatrix}$ can be used to establish that this alone is not enough and hence the determinant established as the balancing factor. (Some calculators will perform matrix algebra and this should be utilized where possible.) Hence $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and if det**A** = 0 the inverse does not exist. Learners should be familiar with writing systems of linear equations in 2 unknowns in matrix form and vice versa. In a system of equations such as $2x-3y=9$, with $4x+\ y=11$ the coefficients of *x* and *y* set up as a 2 by 2 matrix, the unknowns *x* and *y* as a 2 by 1 column matrix (to make the matrices the right shape in order to multiply) and the right hand side of the pair of equations also as a 2 by 1 matrix i.e. $\begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 9 \\ 11 \end{pmatrix}$ The solution can now be facilitated using the algebra of **AX** = **B** leading to $\mathbf{A}^{-1}\mathbf{A}\,\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$ and so $\mathbf{IX} = \mathbf{A}^{-1}\,\mathbf{B}$ or $\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$.	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
1.15 **CCSS: N-VM12**	Work with 2 × 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.	**General guidance** Learners need to grasp the concept of applying various matrices to $\begin{pmatrix} x \\ y \end{pmatrix}$ and determine what happens to this point under various transformational matrices (rotation, reflection and enlargement can be represented by matrix transforms since they all have fixed centers). The *National Stem Centre* activity provides a good visual introduction to this topic. Learners should also be able to apply inverse matrices in a way analogous to solving simultaneous equations, to find a transformational matrix given the vertices of an original polygon and the vertices of the transformed shape. The concept of the determinant can now be built upon, introducing the absolute value of it as the area scale factor of a polygon in the plane and so, under any transform Area of Image = Area of Original × $\lvert \det \mathbf{A} \rvert$ **Teaching activities** Using appropriate software (such as *GeoGebra*), establish the effect of transformational matrices on polygons in the plane. Learners need a great deal of practice, but will soon become familiar with each type of matrix and its resulting effect. Learners need to have access to practice at interpreting the result of applying a matrix as well as actually making the application. Learners should be able to apply the relationship Area of Image = Area of Original × $\lvert \det \mathbf{A} \rvert$ to problem solve including forming equations from given information to solve for unknowns.	Matrices transformations: www.nationalstemcentre.org.uk/elibrary/maths/resource/530/aqa-fp1-matrices-transformations Transformation using matrices: www.mathplanet.com/education/geometry/transformations/transformation-using-matrices How to input matrices using GeoGebra: http://wiki.geogebra.org/en/Matrices Specimen papers are available at http://teachers.cie.org.uk

Unit 2: Algebra

Recommended prior knowledge
Learners will need all of their algebraic skills developed in syllabus 0444 for use here. The emphasis on algebraic processing and reasoning in this syllabus means that the skills in this unit underpin many of the methods used in other units. It is advised that this unit is studied first.

Context
As suggested, it would be advisable to start with this Unit as it builds on 0444 extended. Syllabus ref 2.1 to 2.4 could be grouped and taught together and 2.5 could be taught as a standalone section if required.

Outline
It is strongly suggested that syllabus refs 2.3 and 2.4, division of rational algebraic expressions, are covered prior to 2.1 and 2.2, since many learners will wish to adopt the technique of algebraic long division to solve cubic equations. This will ensure that the manipulative skills required to factorize polynomials and solve cubic equations in 2.1 are covered first and learners will have options regarding method of solution. The sketching of polynomials in 2.2 naturally follows from 2.1. Syllabus ref 2.5, solving simultaneous equations with at least one linear, can be covered at any point and will be good revision of the quadratic skills which underpin many areas of this specification.

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
2.3 **CCSS:** **A-APR4** **A-APR6**	Express simple rational expressions in different forms including writing $\frac{a(x)}{b(x)}$ in the form $q(x)+\frac{r(x)}{b(x)}$ where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, by using inspection or long division and prove polynomial identities.	**General guidance** This technique could be part of the method of solving a more complex problem or it could be a problem in its own right. Learners should be able to perform algebraic long division as well as establish a solution using inspection (with method steps listed). Long division is best considered numerically first and the algebraic steps needed for algebraic long division can then be shown alongside. Learners invariably find the algebraic much easier to learn and apply if they have understood the numerical equivalent, e.g. $187 \div 21$	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		$$\begin{array}{r} 13 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow 21\overline{)287} \leftarrow \text{dividend} \\ \underline{21} \\ 77 \\ \underline{63} \\ 14 \leftarrow \text{remainder} \end{array}$$ It is important for learners to then write the final answer as a mixed number, so $287 \div 21 = 13\tfrac{14}{21}$ Demonstrate with something simple to start with until the method is grasped, so divide a quadratic expression by a linear perhaps, such as $$\begin{array}{r} 2x-5 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow x+4\overline{)2x^2+3x+1} \leftarrow \text{dividend} \\ \underline{2x^2+8x} \downarrow \\ -5x+1 \\ \underline{-5x-20} \\ 21 \leftarrow \text{remainder} \end{array}$$ Now learners must compose the answer as the equivalent of the mixed number so $(2x^2+3x+1)\div(x+4)=2x-5+\dfrac{21}{x+4}$ (A PowerPoint presentation or video, such as the Khan academy video, would be useful for learners here as the method is longwinded to write down in steps and visual presentation would aide later revision). Once this has been mastered move on to dividing polynomials of higher order. Purplemath provides lots of good examples. Note: The remainder theorem with constant remainder is covered in Syllabus ref 2.1 (below) and this could be used to develop the technique	Algebraic long division: www.khanacademy.org/math/algebra/polynomials/v/dividing-polynomials-with-remainders Polynomial long division: www.purplemath.com/modules/polydiv3.htm

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		of inspection as the initial principles are the same and so the introduction of this method could be deferred until after 2.1 has been considered. However, 'inspection' using comparison of coefficients is a sensible approach and uses the fact that the quotient will also be a polynomial. Start with $187 \div 21$ and show that it can also be written as $21 \times 13 + 14 = 187$ $\uparrow$ divisor, $\uparrow$ quotient, $\uparrow$ remainder, $\uparrow$ dividend Now set up the same scenario for the algebraic division previously done by long division, so we have for example, $(x+4)(Ax+B)+R=2x^2+3x+1$ By inspection we see that A = 2. $(x+4)(2x+B)+R=2x^2+3x+1$ $2x^2 \quad +Bx$ $\quad +8x \quad +4B+R$ $2x^2+(B+8x)+(4B+R)=2x^2+3x+1$ Comparing x terms we see that B + 8 = 3, so B = –5 and comparing constants we see that 4B + R = 1, so –20 + R = 1, R = 21. And so we have the result $(x+4)(2x-5)+21=2x^2+3x+1$ which, when we divide through by x + 4 gives us $(2x^2+3x+1)\div(x+4)=2x-5+\frac{21}{x+4}$ as before.	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Teaching activities Activities based on these methods should be straightforward to set up, but lots of practice will be necessary for learners to grasp and retain the skill. The National Stem Centre worksheet provides good reference notes and some questions with answers at the end.	Specimen Paper 2 Question 1 Polynomial division: www.nationalstemcentre.org.uk/elibrary/file/17916/mc-TY-polydiv-2009-1.pdf
2.4 **CCSS:** **A-APR1** **A-APR7**	Add, subtract, multiply, and divide polynomial and rational expressions.	**Notes and exemplars** Understand that polynomials and rational expressions form a system analogous to the integers, namely, they are closed under the operation of addition, subtraction, and multiplication; add, subtract, and multiply polynomials and rational expressions. **General guidance** The notes and exemplars here indicate that learners should be aware that when addition/subtraction and multiplication are applied to polynomials the result is a polynomial and when applied to rational expressions the result is a rational expression. Arithmetic of rational expressions is covered in 0444 extended syllabus and therefore the skills established there should simply need revisiting here. These skills will generally form part of methods needed to access other areas of the syllabus, such as 2.1 and 2.5.	
2.1 **CCSS:** **A-APR2**	Know and use the remainder and factor theorems to find factors of polynomials and solve cubic equations.	**General guidance** Remind learners of quadratic factorizing first and show how knowing the roots or zeros of the expression enable the factors to be found. Once learners have seen this for quadratics extend the concept to cubics. Multiply three linear factors to show that the result is a cubic expression. Show that knowing a zero again enables a factor to be determined. Extending this beyond cubics we have a more formal definition of the factor theorem as **If a polynomial expression, f(x), is such that**	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		f(a) = 0, then (x – a) is a factor and vice versa. Consider methods to factorize a given cubic either after one linear factor has been established or given. Various methods of finding a quadratic factor are sensible – comparing coefficients is a useful method that most learners will grasp well; algebraic long division is often another method that learners grasp well – although sign errors are perhaps more frequent with this technique; synthetic division using zeros rather than factors is also a useful method (*purplemath* has a useful explanation of the method). Learners should be allowed to adopt whichever method they feel most comfortable with. Learners who attempt to factorize purely by inspection, with no method steps being shown, can be prone to errors, so this should not be promoted as a method. For the remainder theorem, the following approach is useful: We have seen with the factor theorem that, if (x – a) is a factor of f(x) (a cubic equation or expression) then f(x) = (x – a) some quadratic If, however, (x – a) is NOT a factor of f(x), then f(x) = (x – a) (some quadratic) + some remainder Calling the remainder R: f(x) = (x – a) (some quadratic) + R However, if we now put x = a we have f(a) = (a – a) (some quadratic) + R So the first bit has become zero and we are just left with f(a) = R and hence the remainder is actually f(a). **If x – a is not a factor, then f(a) = R, where R is the remainder.** These processes can be used to solve cubic equations by factorizing, once the factors have been found. **Teaching activities** Practice finding zeros and writing down the factors from those zeros. Start with basic skills and build so for example, Exercise 2e i.e. in *Pure*	Synthetic division – the process: www.purplemath.com/modules/synthdiv.htm Specimen Paper 2 Question 12 *Pure Mathematics 1*, Chapter 2 (page 33 to 35) Exercise 2e Factoring polynomials: www.purplemath.com/modules/so

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Mathematics 1 gives some good basic practice in this area. www.purplemath.com has an example of factorizing a quintic expression, which could be looked at for extension beyond cubics. *Paul's Online Math Notes* (*tutorial.math.lamar.edu*) are useful for the basic concepts and if one scrolls down to the section on **Factoring Polynomials with Degree Greater than 2.** There are some useful examples of factorizing quartics, which again may be useful for extension.	lvpoly.htm Factoring polynomials: http://tutorial.math.lamar.edu/Classes/Alg/Factoring.aspx
2.2 **CCSS:** **A-APR3** **N-CN9**	Identify zeros of polynomials when suitable factorizations are available, use the zeros to construct a rough graph of the function defined by the polynomial, and know the Fundamental Theorem of Algebra.	**General guidance** The activity from the national stem center is a useful way to help learners understand the relevance of factoring a polynomial to sketching its graph. The *GeoGebra* Polynomial Grapher from *GeoGebraTube* is a useful replacement for the mini-whiteboards mentioned in the stem center activity. The ideas outlined for solving cubic equations are essentially what are required here for finding the zeros or roots of a polynomial and linking that to *x*-intercepts on a graph or sketch. A sketch or rough graph should have the correct shape for the particular polynomial under consideration as well as the *x* and *y*-intercepts all marked as appropriate. Quadratic and Cubic graphs are considered in 0444 extended syllabus and so should not be new to learners. They will need to explore the shapes of polynomials of higher order and can use the *GeoGebra* grapher mentioned above to plot up to and including quintic polynomials. Correct shape of higher order polynomials can be established through considering end behavior, location of zeros and testing signs to determine location/interpolation. The Fundamental Theorem of Algebra can be verified, generalized and thus established in the realm of Real numbers by establishing a pattern via examples of perhaps up to order 5. *GeoGebra* again would be a useful tool for doing this.	Factorising cubics: www.nationalstemcentre.org.uk/elibrary/search?term=factorising+cubics&order=score www.geogebratube.org/material/show/id/12541

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<tr><th>Syllabus ref and CCSS</th><th>Learning objectives</th><th>Suggested teaching activities</th><th>Learning resources</th></tr>
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Note: If complex number has been covered previously, the Fundamental Theorem can be completely established; if not, it can be referred to later once complex values have been introduced.

Teaching activity
Once learners have a good grasp of factoring polynomials and sketching the graphs using this information, activities can be utilized that practice both skills. Exercise 11g in *Pure Mathematics 1* gives good practice at sketching cubics from factorized an unfactorized form.

The *GeoGebra* activity is useful for setting up activities for learners to consider the end behavior and zeros of particular equations which they could be given to investigate if technology were available.

Activities could be given considering the sign of *y*-coordinates for points between the zeros and a table mapping the signs drawn up.
Use the *x*-intercepts of a polynomial function and the sign table to construct a rough graph of the function. e.g.
$y = x^3 - 13x + 12$ has roots or zeros at $x = -4$, 1 and 3 and a sign table could be

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<tr><td>-5</td><td>-4</td><td>-3</td><td>1</td><td>2</td><td>3</td><td></td></tr>
<tr><td>-ve</td><td></td><td></td><td></td><td>-ve</td><td></td><td></td></tr>
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and the basic shape is established by the position of the entries in the table.
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Pure Mathematics 1, Chapter 11 (page 392 to 396) Exercise 11g

www.geogebratube.org/student/m10583

www.autograph-math.com/
Autograph (graphing software) is also an exceptionally useful tool
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<tr><td>2.5

CCSS: A-REI7</td><td>Solve simultaneous equations in two unknowns with at least one linear equation.</td><td>

Notes and exemplars
e.g. find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$

General guidance
Learners should already be familiar with the concept of solving two linear equations and this is simply an extension of those ideas. Looking at the
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Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		example given in the Notes and exemplars graphically, learners will be able to see that it is possible for there to be varying numbers of solutions, depending on the number of intersections of the line and curve under consideration. The *GeoGebraTube* activity gives the opportunity for learners to see what happens as different lines and curves are drawn and see exactly what that means in terms of the number of simultaneous solutions of their equations. Learners should also appreciate what the number of intersections of their line and curve means geometrically – so, for example, that a line which intersects a circle in two points must be a chord or a line which intersects at one point only is a tangent. **Teaching activities** The activity in the *TES* gives a table with structured step by step headings which can be developed and added to for use in the classroom to help learners focus on the steps required to complete the task at hand (the second worksheet under the heading "for pupils".	www.geogebratube.org/student/m7323 Simultaneous equations: www.tes.co.uk/teaching-resource/Simultaneous-Equations-6146699/ Specimen papers are available at http://teachers.cie.org.uk

Unit 3: Functions

Recommended prior knowledge
Learners will be expected to be familiar with all syllabus sections referencing functions in 0444 extended syllabus. It is also advisable that learners have covered Unit 2 (Algebra) prior to this unit in order that the manipulative skills required are established and well-practiced in advance of this unit. Since logarithms are introduced in this unit, learners should be well versed in the laws of exponents.

Context
This unit contains three subsections, which map to the syllabus as subsection A (Functions), subsection B (Sequences and Building Functions) subsection C (Logarithmic and Exponential Functions). It is possible that trigonometric functions will be used in conjunction with the syllabus reference detailed in this unit and for this reason it would be advisable to study this unit prior to studying Unit 7 (Trigonometry), although learners should have enough knowledge of trigonometric functions from 0444 extended syllabus to cover the elements of trigonometric functions required here.

Outline
This unit takes the skills that have been acquired in syllabus 0444 and builds on them so that learners are able to handle a variety of functions, including logarithmic and exponential functions in more challenging settings. The key functional skills and concepts learners will have met in the 0444 extended syllabus will need to be revisited and formalized with more rigor here.

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
Subsection A (Functions)			
3.1 **CCSS: F-IF1**	Understand the terms: function, domain, range (image set), one-one function, inverse function and composition of functions.	**General guidance** The concepts of function, domain, range, one to one, inverse and composite functions should have all been met in syllabus 0444 extended syllabus, however they will need formalizing in the more challenging settings that learners will encounter in this specification. The National Stem Centre pdf document 'Introduction to functions' is a useful reminder to learners about the structure of functions and the idea of domain and range in the first five pages.	Introductions to functions: www.nationalstemcentre.org.uk/elibrary/file/17986/mc-TY-introfns-2009-1.pdf

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
3.2 **CCSS:** **F-IF2**	Use the notation f(x) = sin x, f: $x \mapsto$ lg x, (x > 0), $f^{-1}(x)$ and $f^2(x)$ [= f(f(x))].	**General guidance** Again, the notation here should have all been met in 0444 extended syllabus, however they will need recapping ready for the more challenging settings that learners will encounter in this specification.	
3.4 **CCSS:** **F-BF4** **F-TF6**	Explain in words why a given function is a function or why it does not have an inverse and produce an invertible function from a non-invertible function by restricting the domain.	**Notes and exemplars** E.g. understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.	Inverse functions: www.nationalstemcentre.org.uk/elibrary/file/17944/mc-TY-inverse-2009-1.pdf
3.7 **CCSS:** **A-REI11** **F-IF4** **F-IF7**	Graph functions and show key features of the graph, including understanding points of intersection.	**Notes and exemplars** To include linear, quadratic, square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. **General guidance** The *National Stem Centre* pdf document provides a good explanation of why a function is a function using a graphical approach. It also gives some excellent examples of graphs of functions, including rational functions. **Teaching activity** The same National Stem Centre pdf document has a useful exercise with solutions at the end. It is essential that learners have plenty of practice with these skills.	National Stem Centre: www.nationalstemcentre.org.uk/elibrary/file/17948/mc-TY-trig-2009-1.pdf www.nationalstemcentre.org.uk/elibrary/file/17986/mc-TY-introfns-2009-1.pdf

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
3.5 **CCSS: F-BF4**	Find the inverse of a one-one function and form composite functions, including verifying by composition that one function is the inverse of another.	**General guidance** **Composite function** • Forming the composite function gf means put the function f into the function g, wherever *x* was in g before. $\mathrm{f}(x)=\frac{1}{2}x^2$ $\mathrm{g}(x)=2x+3$ $\mathrm{gf}(x)=\mathrm{g}\left(\frac{1}{2}x^2\right)$ $=2\left(\frac{1}{2}x^2\right)+3$ $=x^2+3$ • Composite functions like gf only exist if the range of values coming out of f is allowed as input for the function g. • The domain of f may need to be restricted for gf to exist. • $\text{Domain}(\mathrm{gf})\subseteq\text{Domain}(\mathrm{f})$ $\text{Range}(\mathrm{gf})\subseteq\text{Range}(\mathrm{g})$ • Learners should verify by example that gf is not usually the same function as fg. The *RISP* activity in the *TES* teaching resources is a great way to investigate this with learners to reinforce this concept. **Inverse function** • f^{-1} exists if the function is one to one (so one input value gives only one output value) If $y=\mathrm{f}(x)$ then $x=\mathrm{f}^{-1}(y)$, so to find the inverse function • Solve for *x* so that *x* = ……….. (BEWARE, when taking e.g. a $\sqrt{\ }$, you must have $\pm\sqrt{\ }$ and in order to choose which root to take (positive or negative) look at the domain of the original function – the *x* value must fit it.) • $x=\mathrm{f}^{-1}(y)$, so now the inverse function has been found but with *y* as the input letter	TES: RISP www.tes.co.uk/teaching-resource/When-does-fg-equals-gf-Risp-18-6056316/

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		• The input letter is usually *x*, so change the input letter to *x* as per usual. • $\text{Domain (f)} = \text{Range}(\text{f}^{-1})$ $\text{Range (f)} = \text{Domain}(\text{f}^{-1})$ The *Khan Academy* examples are excellent for demonstrating finding inverse functions, being careful of the original domain and also demonstrating why a function and its inverse are reflections in the line $y = x$. • Learners should be able to verify that taking ff^{-1} and also taking f^{-1}f should result in *x*, also using this as a test to determine an inverse relationship between two functions. *Purplemath* has useful examples and explanations.	Functions inverse: www.khanacademy.org/math/algebra/algebra-functions/v/function-inverses-example-2 Proving that two functions are inverses of each other: www.purplemath.com/modules/invrsfcn7.htm
3.6 **CCSS: F-BF4**	Use sketch graphs to show the relationship between a function and its inverse.	The *Khan Academy* examples are excellent for demonstrating finding inverse functions, being careful of the original domain and also demonstrating why a function and its inverse are reflections in the line $y = x$. • Graphically, f and f^{-1} are reflections of each other in the line $y = x$ (so if the point (2, 4) lies on f, the point (4, 2) will lie on the inverse function). **Teaching activities** The *National Stem Centre* worksheet gives lots of practice (with answers) at finding inverse functions, restricting domains (including trigonometric functions) as well as graphing functions and their inverses.	Khan Academy: www.khanacademy.org/math/algebra/algebra-functions/v/function-inverses-example-3 Specimen Paper 1 Question 14 National Stem Centre: www.nationalstemcentre.org.uk/elibrary/file/17944/mc-TY-inverse-2009-1.pdf
3.3 **CCSS: F-IF7**	Understand the relationship between $y = \text{f}(x)$ and $y = \|\text{f}(x)\|$, where f(*x*) may be linear, quadratic, or trigonometric.	**General guidance** It is expected that this will be undertaken with a graphical approach. Learners should both be able to draw the graph of $y = \|\text{f}(x)\|$ and find the equation of such a function given the graph. Lots of work can be done using *GeoGebra* to consider graphs such as	Specimen Paper 2 Question 11

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		$y = x^2$ and $y = \|x^2\|$ and debate can arise from such cases. It will be easier to start with straight lines here and consider what is useful information to establish the relationship between the distinct sections of $y = f(x)$ and $y = \|f(x)\|$ – i.e. the x intercept is primarily useful and the y-intercept which is secondarily useful in sketching the graph of the absolute value of the function. The *GeoGebraTube* activity is a simple way to introduce this concept – starting with $y = 2x - 1$ and then allowing the teacher or learner to adjust the gradient and intercept to consider various linear functions. The *Purplemath* web page is good for encouraging learners to carry out the manual process of drawing graphs of such functions. It also moves on to give an example of what the absolute values of a quadratic function would look like, which is the natural next step. Learners of suitable ability level could progress from the purely graphical approach into algebraic processes and link the results they have discovered through their graphical investigations to the algebra e.g. the graph of $y = \|2x - 1\|$ could be used to solve $\|2x - 1\| = 5$ both graphically and algebraically. **Teaching activities** Allow learners to use technology (graphical calculators, appropriate software such as *GeoGebra*) to solve equations involving one or more absolute value graphically and encourage them to confirm their results using algebra. Start with straight lines and build to perhaps one linear and one quadratic or trigonometric (absolute value of either). As an extension activity, once the basic skills have been mastered similar work investigating the solutions of inequalities involving absolute values of functions can be carried out, graphically.	www.geogebratube.org/material/show/id/5538 Graphing absolute-value functions: www.purplemath.com/modules/graphabs.htm

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
3.8 **CCSS:** **F-BF3** **F-LE3**	Recognize even and odd functions from their graphs and algebraic expressions.	**General guidance** Learners should be able to recognize, simply from the type of neither symmetry required, from the graph of a function whether that function is classified as even, odd or neither even nor odd. Again, once this idea has been considered graphically, algebraic exploration can be undertaken. The *Khan Academy* videos are very good explanations of this topic graphically – there is a question posed in the first video about the possible connection between odd and even functions and odd and even numbers and this is resolved in the second video. The algebraic test for odd and even functions is touched upon and thus this can then be explored further. Even functions being such that $f(-x) = f(x)$ and Odd functions being such that $f(-x) = -f(x)$ The *Purplemath* webpage offers good technique for establishing algebraically whether a function under consideration is odd, even or neither. **Teaching activities** *Khan academy* has an interactive activity that learners can use to self assess what they have learned. Learners need to be exposed to a good mix of graphical practice and algebraic practice.	Recognizing odd and even functions: www.khanacademy.org/math/algebra/algebra-functions/v/recognizing-odd-and-even-functions Function notation: www.purplemath.com/modules/fcnnot3.htm Practicing even and odd functions: www.khanacademy.org/math/algebra/algebra-functions/e/even_and_odd_functions
Subsection B (Sequences and Building Functions)			
3.10 **CCSS:** **F-IF3** **F-LE1**	Recognize that sequences are functions, which may be defined recursively and whose domain Is a subset of the inte.g.ers.	**Notes and exemplars** e.g. the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ for $n \geqslant 1$	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		General guidance This builds on 0444 syllabus ref 2.13. Learners should already be familiar with finding an explicit expression (n^{th} term) using position to term and continuing a sequence by considering the term to term relationship. Here they formalize the consideration of the term to term relationship as a recursive or recurrence relation. The basis for a recursive function is • a statement of the initial conditions of the relation (f(0) = f(1) = 1 in the notes and examples) • a definition of the recursive step, stating how to generate the function from its previous values. As well as generating a sequence using a given recursive relation, learners should be able to, for example, find the recursive function given the initial conditions and appropriate terms of the sequence or find the initial conditions, given the recursive step and appropriate terms. **Teaching activities** Plenty of material is available to practice these skills though, in the main, the notation used may need to be adapted from e.g. $u_{n+1} = 3u_n + 1$ to f(n+1) = 3f(n) + 1.	Specimen Paper 2 Question 4
3.9 **CCSS: F-BF1**	Construct a function that describes a relationship between two quantities, including determining an explicit expression or a recursive relation, together with constant multiples, sums, and composites of functions.	**Notes and exemplars** E.g. • Construct a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model. • If T(h) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time. **General guidance** Once learners have mastered the techniques of composing functions and working with recursive functions, they can apply the skills they have learned to practical situations.	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Here, they will need to be able to interpret word problems as functions and show understanding of what they have learned. Learners should understand how patterns and relationships of algebraic representations are used to generalize, communicate, and model situations in mathematics. *Purplemath* gives some simple word problems which are then written up as formal functions and this is a good way to introduce and develop this area of the syllabus. The *ugrad* website also gives a simple example which will help learners understand how functions can be related by composition.	Compositions of functions: www.purplemath.com/modules/fcncomp5.htm www.ugrad.math.ubc.ca/coursedoc/math100/notes/zoo/composite.html
Subsection C (Logarithmic and Exponential Functions)			
3.11 **CCSS:** **F-IF7** **F-BF5**	Understand and know simple properties and graphs of the logarithmic and exponential functions including ln x and e^x (series expansions are not required) including interpreting the parameters in a linear or exponential function in terms of a context.	**General guidance** Learners are familiar with exponential functions from 0444 extended syllabus. They will now need to be introduced to the exponential function, e^x. Since series expansions are not required and learners have pre-knowledge of the graphs of exponential functions in general and estimating the gradient of a function by drawing a tangent to the function at a point, e^x can be introduced as the exponential function whose gradient at the point (0, 1) is equal to 1. The *GeoGebraTube* activity may be a good introduction to this idea, though e^x is not plotted, it can be seen that as a^x becomes closer to e^x, the gradient of the tangent gets closer to 1. The *Autograph* activity is similar and may be preferable. Graphs of 2^x , e^x and 3^x can all be plotted using appropriate software such as *GeoGebra*, and the numerical value of e be established. For a good group of learners the *Intuitive explanation of e as the unit rate of growth* gives a really good insight.	www.geogebratube.org/student/m4976 Autograph activity – Discovering e: www.tes.co.uk/teaching-resource/Autograph-Activity-Discovering-e-6263598/

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Logarithms will be a new concept and need careful introduction as learners are easily confused if the definition is not carefully laid down at the start. If $y = a^x$, then we define *x* to be the logarithm of *y* to the base of *a*. i.e. the log is the power to which *a* has to be raised to achieve *y*. If $y = a^x$ then $\log_a y = x$ a > 0. Introduce the idea of a common log and logs to various numerical bases to start with and give plenty of examples so that learners start to become comfortable with the concept. It would be useful for some of this work to be carried out without using a calculator. E.g. Find the value of $\log_{10} 1000$ by asking the question "To what exponent must you raise 10 to achieve 1000?" Natural logs can then be introduced as the solution to e.g. Find *x* such that $e^x = 0.59$. Introduce the notation $\log_e y = \ln y$ and start to look at using the calculator to evaluate expressions, leading to establishing $\log_a 1 = 0$ (and so ln 1 = 0) and $\log_a a = 1$ (and so ln e = 1). The natural progression in teaching would be to explore the idea of ln*x* and e^x as inverse functions. The *National Stem Centre* worksheet has a good teaching progression. **Teaching activity** The *TES* activity, which is a set of dominoes linking inverses of exponential and logarithmic functions is a useful tool to make connections between the two. The exercise in *Pure Mathematics 1* offers lots of practice and should help to establish a good and clear understanding of what a logarithm actually is.	An intuitive guide to exponential functions and e: http://betterexplained.com/articles/an-intuitive-guide-to-exponential-functions-e/ Exponential and logarithm functions: www.nationalstemcentre.org.uk/elibrary/file/17942/mc-TY-explogfns-2009-1.pdf Inverses dominoes: www.tes.co.uk/ResourceDetail.aspx?storyCode=6033007 *Pure Mathematics 1* Chapter 2 Exercise 2c

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
3.12 **CCSS: F-LE4**	Know and use the laws of logarithms (including change of base of logarithms).	**Notes and exemplars** Simplify expressions and solve simple equations involving logarithms. **Log Law 1 The Addition Law** The proof of $\log_a x + \log_a y = \log_a(xy)$ could be demonstrated using $x = a^m$, $y = a^n$ and $xy = a^{m+n}$, the laws of exponents and the basic definition of a logarithm. The proof need not be memorized but must be given to enable learners develop understanding. **Log Law 2 The Subtraction Law** The proof of $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ could be demonstrated using $x = a^m$, $y = a^n$ and $\frac{x}{y} = a^{m-n}$, the laws of exponents and the basic definition of a logarithm. **Log Law 3 The Exponent Law** The proof of $\log_a(x^n) = n\log_a x$ could be demonstrated using $x = a^m$, $x^n = (a^m)^n = a^{mn}$, the laws of exponents and the basic definition of a logarithm. Even though it technically is not a law of logarithms, change of base of logs could be given as **Log Law 4 Change of Base** Let $x = a^m$ then $\log_a x = m$ and since for all real numbers we should be able to write $a = b^c$ so that $\log_b a = c$ we have $x = (b^c)^m = b^{cm}$ so $\log_b x = cm$ and $\frac{\log_b x}{c} = m$ and the result $\log_a x = \frac{\log_b x}{\log_b a}$ follows.	Specimen Paper 2 Question 6

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		A good group of learners could be allowed to develop Laws 1 to 3 for themselves, given the simple information detailed above. Working in teams, they could each be given a different law to develop and present to the group as a whole. **Teaching activities** It cannot be over-emphasized that practice will be the key to success here. Once they have mastered the basic laws they should have the opportunity to revisit the skill often. Short questions could easily be included in starters to lessons and included in quizzes and mixed homeworks. The *Rich Starting Points for A Level Maths* activity is a tough but rewarding one that can be used to revise logs and their properties. The exercise in *Pure Mathematics 1* provides plenty of practice in using log laws.	Building log equations: www.s253053503.websitehome.co.uk/risps/risp31.html *Pure Mathematics 1* Chapter 2 Exercise 2d Q1 - 4
3.13 **CCSS:** **F-BF4** **F-BF5** **F-LE4**	Solve equations of the form $a^x = b$, including use of logarithms to base 2, 10, or e.	**Notes and exemplars** e.g. • solve $5^x = 7$ • solve $2e^{3x} = 12$ **General guidance** The solution of the equations given in the notes and examples can be achieved either by using the basic definition of a log and then using a scientific calculator to solve to any base. If a calculator does not have this capability then learners could apply Law 4 – changing the base. Alternatively learners could apply Law 3 – and bring down the exponent. In either case, learners should be encouraged to isolate the exponential function prior to applying any log laws, so simplify $2e^{3x} = 12$ to $e^{3x} = 6$ as this reduces the number of errors learners make in examinations when asked to solve this type of equation.	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Teaching activities The *nrich* activity *Log Attack* is an interesting challenge; similarly the How Many Solutions? activity. The *Pure Mathematics 1* exercise provides more opportunities at practice by solving equations who roots are the solutions to equations of the form $a^x = b$.	Log attack problem: http://nrich.maths.org/5831 How many solutions?: http://nrich.maths.org/334 *Pure Mathematics 1* Chapter 2 Exercise 2d Q5
3.14 **CCSS:** **F-LE1**	Distinguish between situations that can be modelled with linear functions and with exponential functions.	**General guidance** Learners should recognize that situations that can be modeled with a linear function can be identified by having a constant rate of change, whereas situations where an exponential model is appropriate have a rate of change that is a constant percent rate. This is an extension of work already covered in 0444 syllabus and it is anticipated that it would be included as a component part of a greater problem, rather than a topic for assessment in its own right. This is a pre-calculus skill.	Specimen papers are available at http://teachers.cie.org.uk

Unit 4: Geometry

Recommended prior knowledge
All the material in 0444 extended syllabus should have been covered prior to this unit being undertaken. Circle theorems do not form part of this unit, but learners will be expected to know and use the theorems they have encountered with respect to lines and angles, triangles (including the Pythagorean Theorem) and parallelograms.

Context
This unit could be taught at any point during the course.

Outline
The key theme for this unit is congruence. Learners will have met the idea of congruence in 0444 extended syllabus, but not used it to solve problems. The rudiments of congruence in terms of rigid motions can be investigated and developed to the extent where learners are capable of solving more complex problems through formal geometric proofs.

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
4.2 CCSS: G-CO8	Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motion.	The expectation here is for learners to understand that rigid motions are at the foundation of the definition of congruence. Learners reason from the basic properties of rigid motions (that they preserve distance and angle), which are assumed without proof. Rigid motions and their assumed properties can be used to establish the usual triangle congruence criteria, which can then be used to prove other theorems. Learners need to explore what basic criteria must be fulfilled in order to prove beyond doubt that two triangles are congruent. As well as ASA, SAS and SSS, learners could be encouraged to investigate right-angled triangles to arrive at the added criteria HL (or HLR or RHS).Demonstrate visually why some conditions like SSA or AAA are not sufficient to show congruence.	 Congruent triangles: www.mathopenref.com/congruenttriangles.html

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Give lots of straightforward simple practice here considering two triangles and establishing their congruency. The *mathwarehouse* activity gives a good basic introduction and should help to reinforce the criteria mentioned. The two column reasoning approach is a good one and reminds learners that a reason at each step in their proof must be given and reduces the number of erroneous assumptions made. For example: Statements | Reasons	Proving triangle are congruent: www.mathwarehouse.com/classroom/worksheets/congruent_triangles/Triangle_proof_ASA-SAS.pdf
4.1 **CCSS:** **G-CO6** **G-CO7**	Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if, and only if, corresponding pairs of sides and corresponding pairs of angles are congruent.	Using the principle of superposition, learners should be to list a sequence of rigid motions which map one triangle to another, to show that that they are congruent. The rigid motions in the sequence should be clearly and accurately defined in this type of question. Ensure that learners use a variety of tools to explore and perform simple, multi-step, and composite rotations, reflections, and translations. The acronym CPCTC (corresponding parts of congruent triangles are congruent) is a useful mechanism here. This will form the basis for many of the geometric proofs learners will be expected to complete. The *mathwarehouse* activites are interactive and take learners step by step through some proofs using CPCTC. These may be looked at after 4.2 is also covered if desired and will probably have more impact then. They could be done as independent exercises or given to learners in small groups or demonstrated to the class as a whole to promote good practice. The *letspracticegeometry* worksheet gives ample written practice at the skill or presenting a reasonable proof.	*Geometry Essentials for Dummies* Chapter 5 Congruent triangle are congruent: www.mathwarehouse.com/geometry/congruent_triangles/congruent-parts-CPCTC.php Proofs involving CPCTC: www.letspracticegeometry.com/wp-content/uploads/2011/11/proofs-involving-CPCTC.pdf

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		The *glencoe* web link is to a set of interactive activities that require learners to complete the steps in a proof from a selection of statements and reasons, to work as teams and there is also a quiz. An excellent resource to help learners grasp the fundamentals of what is necessary here.	Proofs and congruent triangles: www.glencoe.com/sites/
4.3 **CCSS:** **G-CO9** **G-CO10** **G-CO11** **G-SRT4** **G-SRT5**	Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.	**Notes and exemplars** Candidates will be expected to know and use the following theorems in their proofs: **Lines and angles**: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints. **Triangles**: measure of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segments joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point; a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity. **Parallelograms**: opposite sides are congruent; opposite angles are congruent; the diagonals of a parallelogram bisect each other and conversely, rectangles are parallelograms with congruent diagonals. Learners should now have enough knowledge at their disposal to apply their skills to more complex problem solving. All of the above theorems could be included as a necessary part of a geometric proof set at this level and thus it is essential that plenty of practice at decomposing more complex problems is given. As the key features of this syllabus ref are congruence and similarity, it is sensible to focus on those first and determine, for example, which two triangles to prove congruent or which two shapes to prove similar. Once a learner has established that, then	Specimen Paper 1 Question 6 available at http://teachers.cie.org.uk

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		the given information should be considered and the steps of the proof constructed. The necessity to give reasons at every step should be emphasized in all the work for this unit. The *Making Sixty nrich* activity can be used with any group to start them thinking logically about problem solving using congruency. The other two activities here from *nrich* could be used as an extension idea with a good group.	 Making sixty: http://nrich.maths.org/6355 Bang's theorem: http://nrich.maths.org/272 Napkin – teacher's notes: http://nrich.maths.org/700/note

Unit 5: Transformations and vectors

Recommended prior knowledge
All relevant sections of 0444 extended syllabus should have been covered. Learners will also need trigonometric skills applied to both right-angled and non-right-angled triangles and also to be able to apply the Pythagorean Theorem.

Context
It is advised that this unit is studied prior to Unit 1 subsections B and C, since learners have prior knowledge of transformations and vectors which can be built upon to introduce complex number and matrices. In the first instance, a basic understanding of vectors comes from differentiating between scalar and vector quantities. The movement from an original position to a final position of a translation represents a vector. This understanding may need formalizing at this level. Vectors have been introduced geometrically in a plane in 0444 extended and then algebraically. The application to many areas of science, engineering and applied mathematics should become apparent through the problem solving activities undertaken.

Outline
The vector spaces considered here are two dimensional, not three dimensional. There are two main themes within this unit – theoretical applications of vector processes, which may be geometric or algebraic in nature and practical applications involving, for example, velocity vectors.

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
5.1 **CCSS:** **N-VM1** **N-VM2** **N-VM4** **N-VM5**	Use vectors in any form, e.g. $\begin{pmatrix} x \\ y \end{pmatrix}$, $\overrightarrow{AB}$, $\mathbf{a}$, $x\mathbf{i} + y\mathbf{j}$.	**General guidance** This needs practicing throughout the unit rather than being treated as a separate component. However it is necessary to be rigorous with learner use of symbols for vectors and to understand the different forms used in text and handwritten mathematics. The Cartesian component form should be the only new presentation.	
5.2 **CCSS:** **N-VM1**	Know and use position vectors and unit vectors.	**General guidance** Learners will already be familiar with position vectors from 0444 extended syllabus. They will also already know how to calculate the magnitude of a vector. Here these skills are developed into	Specimen Paper 2 Question 2

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		consideration of unit vectors also. *GeoGebra* or similar software could be used to establish geometrically the unit vector in a given direction and then similarity could be used to establish the result: For any non-unit vector, **a**, the unit vector in the same direction is given by $\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}$, where $\lvert\mathbf{a}\rvert$ is the magnitude of **a** found using the Pythagorean Theorem. Once the concept of a unit vector has been established, the Cartesian (or rectangular) components can be introduced with **i** as the unit vector in the positive direction of the *x*-axis and **j** as the unit vector in the positive direction of the *y*-axis. These should be linked to column vectors and learners should be able to transfer easily between the two forms. The *TES* resource is a PowerPoint introducing Cartesian components and giving some examples for learners to practice.	Cartesian components: www.tes.co.uk/teaching-resource/Cartesian-components-3004262/
5.3 **CCSS:** **N-VM1** **N-VM4** **N-VM5**	Find the magnitude and direction of a vector, add and subtract vectors and multiply vectors by scalars; determine the magnitude and direction of the sum of two vectors.	Learners should already have experience in finding the magnitude, adding, subtracting and multiplying a single vector by a scalar using column vectors and geometric reasoning problems. The vectors multiple choice PowerPoint presentation revises some key concepts using geometric reasoning problems and is intended to be used with a voting or multiple choice systems – though it could easily be adapted into a simple group quiz. These skills can be considered using Cartesian components. The type of problem that learners will expect to be able to solve will be more challenging at this level. A basic understanding of vectors comes from differentiating between scalar and vector quantities. The *Khan Academy* video gives a basic introduction to distance/displacement and speed/velocity which could be used as a platform to discuss other types of physical scalar and vector	Vectors multiple choice: www.tes.co.uk/teaching-resource/Vectors-multiple-choice-6161352/ Introduction to vectors and scalars: www.khanacademy.org/science/physics/v/introduction-to-vectors-

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		quantities. Determining the direction of a vector is a new skill as is determining the magnitude and direction of the sum of two vectors. The direction should be calculated using trigonometry (this might involve using the sine or the cosine rule). The expected direction is usually the angle made between the vector and the *x*-axis in a theoretical problem using a Cartesian frame of reference or perhaps a bearing (i.e. measured from north and clockwise) in a practical problem. If the problem does not specify from which point the direction is to be given, learners should always make that clear in their solution. At this level, it is expected that direction will be given as an accurate value, so a response of “approximately north west” would not be considered correct. The *mathwarehouse* activity is a good introduction to the direction of a vector. If it is done as a group activity, it offers a really good opportunity to discuss from what point the direction should be given and the issues that arise if an angle is given with no reference to from which point the angle has been calculated or measured. The *GeoGebraTube* activity shows the result of adding two vectors and will be a useful starting point for considering the magnitude and direction of the sum of two vectors.	and-scalars Magnitude & direction of a vector: www.mathwarehouse.com/vectors/ www.geogebratube.org/student/m12566
5.4 **CCSS: N-VM3**	Compose and resolve velocities.	The same *mathwarehouse* activity mentioned above in 5.3 has questions on resolving vectors given direction and magnitude into Cartesian component form. The final components are given as coordinates and this could be developed as an opportunity to practice writing vectors in $x\mathbf{i} + y\mathbf{j}$ form. This could then act as a stepping stone to introduce the concept of resolving a velocity vector. Vector resolution is taking a resultant vector and breaking it down into two or more component vectors. Vector composition is the process of determining a resultant vector by adding two or more vectors together. The new vector is called the	Magnitude & direction of a vector: www.mathwarehouse.com/vectors/

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		resultant vector. If the quantity is a velocity, then the new vector is a resultant velocity. These skills are fundamental to solving problems involving velocity and should be practiced continually to enable learners to have the tools required to tackle the type of practical problem they will face during assessment.	
5.5 **CCSS:** **N-VM3**	Solve problems involving velocity and other quantities that can be represented by vectors and use relative velocity, including solving problems on interception (but not closest approach).	It will be necessary to • relate vectors to bearings. • relate vectors to velocity of planes when affected by crosswinds. • relate vectors to velocity of boats when affected by a current and so on. A simple diagram is always a good idea! However, it must be stressed learners will be expected to calculate rather than use scale drawings to solve such problems. The *teachengineering* activity is a great way to introduce the idea of a boat being steered off course by wind. It could be developed further than it has been by considering on what course the boat should have been set initially to reach the point on the US that it would have reached under totally calm conditions.	Specimen Paper 1 Question 12 TeachEngineering – Vector voyage: www.teachengineering.org/ Specimen papers are available at http://teachers.cie.org.uk

Unit 6: Coordinate geometry

Recommended prior knowledge
It is recommended that learners have completed Unit 2 prior to studying this unit. They will require the Pythagorean Theorem and perhaps solving simultaneous equations for this unit. It is expected that all the algebra and coordinate geometry sections of the syllabus in 0444 extended have been covered prior to studying this unit. Precise definitions of polygons and so on, covered in 0444 extended syllabus 4.1 to 4.4 may also be required here.

Context
This unit has been broken into two subsections, subsection A (The Straight Line and Coordinates) and subsection B (Equations of Conic Sections). It is recommended that learners cover Unit 1 subsection A prior to this unit in order to recap solving radical equations, a skill they are likely to need for Conic Sections. There are natural links between the two subsections, as identified in the notes and examples for syllabus ref 6.3. However, it would be quite possible for learners to study subsection A and come back to subsection B at a later point in time if required. There is plenty of material based on polygons, for example, that can be used to as a basis for problems referenced under syllabus ref 6.3 if that is the case. Syllabus ref 6.1 also links to line of best fit, syllabus ref 9.8, where it will be expected that learners interpret the line in terms of a context. It is sensible to have covered this unit prior to Unit 9. Learners should not be solving problems in this part of the syllabus by scale drawings.

Outline
Subsection A (The Straight Line and Coordinates) has, in the most part been covered in 0444 extended and the first three learning objectives will involve application of skills previously learned. Subsection B (Equations of Conic Sections) is new material and learners will be expected to **apply** all their prior algebraic and co-ordinate geometry skills to solving geometric problems. Problem solving techniques will be very important in this unit and learners should consider something along the lines of:

Given:	What have they told you and why?
Asked for:	What are you trying to find out?
Unknowns:	What do you not know that you need to answer the question?
Strategy:	How are you going to find the unknowns and solve the problem? (solve equations, deductive reasoning, trial & improvement, working back, etc.)
Solve:	Apply your strategy.

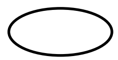
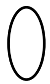
Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
Subsection A (The Straight Line and Coordinates)			
6.1 **CCSS: F-LE5**	Interpret the equation of a straight line graph in the form $y = mx + c$.	**General guidance** Learners will be able to apply their previously acquired skills to solving challenging problems in this syllabus. The parameters (the gradient and intercept) may need interpreting in terms of that context. The *Purplemath* activity gives some good examples of interpreting the general equation of a straight line in context. The *TES* resource is a PowerPoint that could be used as a starter quiz to recap on the reading the gradient and intercept from y = *mx* + *c*. Given the equations of two non-parallel lines, learners should be able find their point of intersection using previously acquired skills.	The meaning of slope and y-intercept: www.purplemath.com/modules/slopyint.htm y=mx + c quick test: www.tes.co.uk/ResourceDetail.aspx?storyCode=6111880
6.2 **CCSS: G-GPE5**	Use slope criteria for parallel and perpendicular lines to solve geometric problems with justification.	Learners should understand that • slopes of parallel lines are equal • slopes of perpendicular lines are opposite reciprocals (i.e., the slopes of perpendicular lines have a product of -1). Learners should be able to find • the equation of a line parallel to a given line that passes through a given point • the equation of a line perpendicular to a given line that passes through a given point. The *TES* resource is a puzzle that can be used to reinforce the skills detailed above. Determining and using the parallel and perpendicular line relationships will most likely form part of the method for a more advanced problem that learners will need to work through with much less structure given to guide them. They will need to link different elements of their knowledge together and use them together as a cohesive whole to present a logical and reasoned argument.	Specimen Paper 1 Question 10 Equations of straight lines: www.tes.co.uk/teaching-resource/Equations-of-straight-lines-6148248/

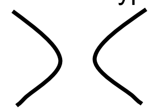
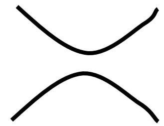
Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
6.3 **CCSS:** **G-GPE1** **G-GPE5**	Solve questions involving midpoint and length of a line.	**Notes and exemplars** e.g. to find the equation of a circle given the endpoints of the diameter **General guidance** The example in the notes and examples would most naturally be covered after syllabus ref 6.5. However, the questions learners should be presented with should not be limited to conic sections and other polygons should be considered also. Learners could investigate whether a shape was a particular quadrilateral for example – e.g. a parallelogram by considering length of side and slope. Another possibility is finding the equation of the line of symmetry of an isosceles triangle, given the coordinates of its vertices.	Specimen Paper 1 Question 9
6.4 **CCSS:** **G-GPE4**	Use coordinates to prove simple geometric properties algebraically.	**Notes and exemplars** e.g. • determine whether a figure defined by four given points in the coordinate plane is a rectangle • determine whether the point (1, √3) lies on the circle which is centered at the origin and passes through the point (0, 2) **General guidance** Again, the example in the notes and examples referencing the circle would most naturally be covered after syllabus ref 6.5. However, the questions learners should be presented with should not be limited to conic sections and other polygons should be considered also. The skills already referenced above i.e. • finding and using slopes including parallel and perpendicular lines • calculating distance between two points, one of which may be a general point (x, y) or equivalent • applying midpoint, Pythagorean Theorem and basic knowledge of polygons – especially those that have perpendicular diagonals such	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		as the kite or rhombus, or parallel sides such as the parallelogram and trapezoid or indeed special triangles • basic properties of regular polygons are all essential prerequisites here. This is a drawing together of all the skills they have acquired and an application of them to situations where they have to construct their geometrically reasoned argument from the basics. Always start with a grid diagram and the geometry and build the algebra from this basis. Learners will have a much better perception if this is the case. Learners could be set investigative tasks such as finding the point of intersection of the diagonals of a parallelogram by solving simultaneous equations and then testing whether or not the diagonals of a parallelogram bisect each other by considering the midpoints of the diagonals.	
Subsection B (Equations of Conic Sections)			
		General guidance If possible use a model of a double cone to illustrate conic sections geometrically as cross sections of a cone. If this is not possible, software such as the wolfram plug-in provide really good insight and an opportunity for learners to see what happens as the double cone is sliced with various planes. The image can be rotated and learners can see a circle become an ellipse, parabola or hyperbola. Start each topic with conics centered at the origin and develop the standard equation of each conic from that, then translate the whole graph to different centers to establish the geometry and general equation for any center. For all the conic sections, it is important not to delve too deeply into the theory at this stage as this is designed to be an introduction into the subject rather than a detailed analysis. The idea here is to make connections between the geometry of the conic section and the algebraic interpretation of that geometry. Demonstrating graphically	http://demonstrations.wolfram.com/ConicSectionsTheDoubleCone/

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		using appropriate software or graphical calculators, as well as the software suggested should provide suitable justification for the processes. Learners will then need to practice the algebra to develop the skills and keep coming back to it over time so that the skills are not neglected and forgotten.	
6.5 **CCSS: G-GPE1**	Derive the equation of a circle given centre and radius using the Pythagorean Theorem; complete the square to find the centre and radius of a circle given by an equation.	**General guidance** **Circle** Learners should learn how to • define a circle in terms of the locus of a point moving (with coordinates (*x*, *y*)) so that it is a constant distance (the radius) from a fixed point (the centre with coordinates (*a*, *b*)). • find the distance between two points defined in terms of their coordinates in order to derive the equation of a circle in the form $(x-a)^2+(y-b)^2=r^2$ ① . This should then be expanded so that learners understand that the general equation can also be written in the form $x^2+y^2+2gx+2fy+c=0$ ② Learners should then be able to complete the square on *x* and on *y* to rewrite equation ② in the form of equation ① in order to easily read off the centre and radius of the circle. It could also be established that, if the circle is given in form ②, the centre is given by (–*g*, –*f*) and radius $\sqrt{g^2+f^2-c}$. Learners should practice the simple skills as outlined in the learning objective and also use the knowledge they have about the equation and geometry of a circle to problem solve. For example, they should be able to • find the equation of a circle given the coordinates of the end point of a diameter • prove, or otherwise, that a line segment between two points in	Specimen Paper 1 Question 1

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		the plane is a diameter of a circle • given the equation of a circle and a point, determine whether the point lies outside, inside, or on the circle • given the equation of a circle and a point on it, find an equation of the line tangent to the circle at that point*. *There is some overlap here between what is expected from this syllabus ref and syllabus ref 6.4. The last example above is perhaps better suited to 6.4, but is included here because of the reference to the equation of the circle. As extensions of the ideas here, learners could consider further problems, such as those determining whether two circles intersect, touch or do not intersect by considering the sums or differences of their radii.	
6.6 **CCSS: G-GPE2**	Derive the equation of a parabola given a focus and directrix.	**General guidance** **Parabola** Learners should learn how to • define a parabola as the path traced out by a point as it moves so that it is an equal distance from a fixed point (the focus) and a given line (the directrix) • derive the equation of a parabola given the focus and directrix using the vertex form of the parabola either $y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$ • identify the vertex, focus, directrix, and axis of symmetry The basic idea is excellently demonstrated in the *GeoGebraTube* resources, one of which enables an horizontal parabola to be drawn from the definition and one which enables a vertical parabola to be drawn. Graphing software or graphical calculators would be excellent tools to use here to consolidate and confirm the skills and processes. Since learners are not expected to know the formula for finding the perpendicular distance from a point to a line, the material under	www.geogebratube.org/student/m7505 www.geogebratube.org/material/show/id/5020 Conics: Paraboloas: www.purplemath.com/modules/pa

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		consideration for this topic is essentially limited to vertical and horizontal directrices. However, learners could derive the perpendicular distance from the focus to the directrix as part of a lengthier, more structured question on coordinate geometry and so it may be possible for the equations of more general parabolas to be established in certain circumstances. *Purplemath* offers a good introduction to this topic, starting with the equation and finding the focus and directrix from it, which engenders understanding and moving on to forming the equation from the focus and directrix. The approach here is to build the equation starting with the distance between the focus and directrix and halving it to find the vertex of the parabola. Learners should also practice finding the distance from a general point to a specific point and using the expression derived to establish the equation of the parabola, given the directrix. This approach builds the equation starting with the distance between the focus and the moving point which traces out the parabola.	rabola.htm Specimen Paper 1 Question 4
6.7 **CCSS: G-GPE3**	Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.	**General guidance** **Ellipse** Learners will need to be able to • find expressions for the distances between the moving point whose path traces out the ellipse and the two foci of an ellipse • given the constant sum between these distances, form and solve a radical equation to derive the equation of the ellipse in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ for  or $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ for  • identify the major and minor axis. The basic idea is excellently demonstrated in the *GeoGebraTube* resources, *an ellipse to be drawn from definition*.	Introduction to the ellipse: www.khanacademy.org/math/algebra/conic-sections/v/conic-sections--intro-to-ellipses Conics: Ellipses: www.purplemath.com/modules/ellipse.htm

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		If preferred, learners can carry out a practical activity using tacks and a length of string. Fix the ends of the string to the two tacks and pin them to a board securely. Place a pencil in the loop of the string and keeping the string taut, trace out the locus of the end of the pencil as it moves. The result should be an ellipse whose greatest distance across should be the same length as the string between the pins. The pins are the foci, the string is the length of the major axis and it can be shown that this is 2*a*. As the pencil moves, the distance between it and the two pins is constant and the result follows. **Hyperbola** Learners will need to be able to • find expressions for the distances between the moving point whose path traces out the hyperbola and the two foci of an hyperbola • given the constant difference between these distances, form and solve a radical equation to derive the equation of the hyperbola in the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ for  or the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ for  • identify the vertices, center, transverse axis, conjugate axis, and asymptotes. The basic idea is excellently demonstrated in the GeoGebraTube resource with an hyperbola being drawn from this definition. Visual presentation of this topic will be central to learners overall understanding of what they are attempting to find when deriving the required equations.	www.geogebratube.org/student/m10565 Conics: Hyperbolas:

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Rectangular Hyperbola This is a special case of a regular hyperbola where the two asymptotes are perpendicular. For this case the equation will be of the form $x^2 - y^2 = a^2$. A particular case of the rectangular hyperbola occurs when the asymptotes are vertical and horizontal – in these circumstances the equation will be $(x-h)(y-k)=m$	www.purplemath.com/modules/hyperbola.htm www.geogebratube.org/student/m10566
		Learners will find the terminology difficult in this subsection, so the correct names for all the key features should be reinforced when possible. Learners might like to undertake research into why each conic section is given its particular name.	Specimen papers are available at http://teachers.cie.org.uk

Unit 7: Trigonometry

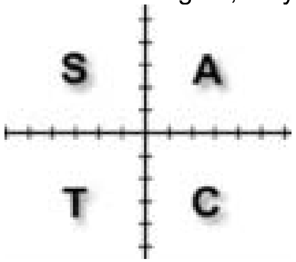
Recommended prior knowledge
Learners should have covered all the trigonometric topics from 0444 extended syllabus. Learners will need algebraic manipulation skills to prove trigonometric identities. They will also need to be well versed in transforming graphs in general as this is a key feature of this unit and the ideas are extended here.

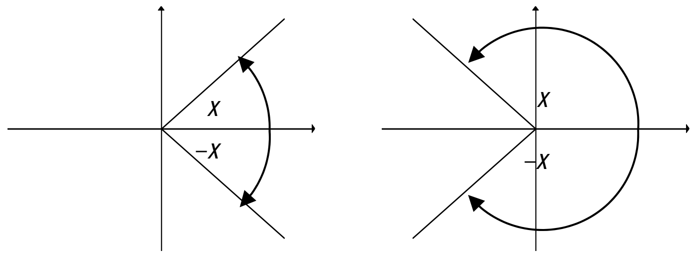
Context
It is advisable that learners to have covered Unit 3 (functions) either prior to this unit or, in part, in conjunction with it. Since trigonometric proofs are introduced here, it is also advisable that learners have studied Unit 4, since the approaches to the structure of a proof should be similar.

Outline
The concept of radian measure is introduced in this unit. This unit builds on the foundations learners have already acquired in trigonometry. They start to go beyond the application of trigonometric ratios to acute angles and develop pre-skills for general solutions of trigonometric equations, by working with angles of any magnitude. Learners develop graphical skills relating to the transformation and application of trigonometric functions, the technique leading to the solution of practical problems modeled by trigonometric functions. Simple trigonometric proofs are also introduced.

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
7.1 **CCSS:** **F-TF1** **G-C5**	Solve problems involving the arc length and sector area of a circle, including knowledge and use of radian measure.	**Notes and exemplars** Derive, using similarity, the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. **General guidance** Emphasize the similarity of all circles. Note that by similarity of sectors with the same central angle, arc lengths are proportional to the radius. Use this as a basis for introducing radian as a unit of measure.	

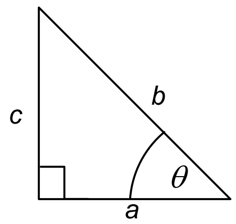
Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Define one radian as the angle subtended at the centre of a circle by an arc of length, *s*, equal to the radius. Investigate with learners how many arcs of length *r* can be cut from the circumference of a circle. Help learners to arrive at the conclusion that 360° = 2π radians. Once the concept of radian measure has been fully understood, the *RISP* activity could be used as a homework activity or perhaps as an in class investigation to be done in groups and the results pooled. Note at this point it should be strongly emphasized that the unit is “radian” not “π radian” as it is a common misconception that all angles measured in this unit are named “π radians”. From this basis and using learners’ prior knowledge of the circumference and area of a circle, learners can derive the results $s = r\theta$ (establishing that θ in radians is the constant of proportionality) and $A = \frac{1}{2} r^2\theta$ and these used to solve problems. Learners must ensure that their calculators are in the correct unit of angular measure appropriate to the problem they are solving. This unit of angular measure should be practiced at all stages through the unit and learners should be able to work in either degrees or radians, recognizing which is appropriate.	Radians and degrees: www.s253053503.websitehome.co.uk/risps/tes-risp-23.pdf Holly problem: http://nrich.maths.org/681 Sectors of circles: http://map.mathshell.org/materials/download.php?fileid=1284
7.2 **CCSS: F-TF2**	Know and use the three trigonometric functions of angles of any magnitude (sine, cosine, tangent).	**General guidance** The *GeoGebraTube* activity presents a neat visual presentation of how the unit circle is linked to the trigonometric functions cosine and sine. Learners need to become familiar with the conventions of a counter clockwise angle being positive and a clockwise angle being negative. The *Haesemathematics* resource is relevant to more than one section of this unit and could be used to extend learners’ perceptions of the trig ratios beyond the usual domains of 0° to 360° or –180° to 180°.	www.geogebratube.org/student/m3342 Advanced trigonometry: www.haesemathematics.com.au/samples/ibmyp5plus-2_18.pdf

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Once learners have grasped the connections between angles of any magnitude and the periodic nature of the trigonometric functions sine, cosine and tangent, they should be introduced to the quadrant diagram  to help them recall which trigonometric ratio is positive in which quadrant. Emphasis should be placed on working from the horizontal axis here. Learners could make up their own mnemonic to recall which quadrant is which. *Worsley School* gives a good explanation of how this works and the *GeoGebraTube* activity demonstrates it very neatly.	www.geogebratube.org/student/m12795 The cast diagram: www.worsleyschool.net/science/files/cast/castdiagram.html
7.3 **CCSS: F-TF3**	Determine geometrically the values of sine, cosine, tangent for $\frac{\pi}{3}$ and $\frac{\pi}{4}$ and $\frac{\pi}{6}$; express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.	The special triangles, right-angled isosceles with equal unit sides and equilateral with sides 2 are the usual method for determining values of trigonometric ratios relating to the angles mentioned here. Since learners are expected to know the exact values for the trigonometric ratios of 0°, 30°, 45°, 60°, 90° for 0444 extended syllabus, this should be nothing more than a recapping of that using radians rather than degrees. The second element of this learning objective could be introduced when considering the unit circle and using angles of any magnitude, the previous learning objective.	Basic triangle values: www.purplemath.com/modules/trig.htm

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		 Considering the above graphs, should lead learners to conclude that cos$x$ = cos($2\pi - x$) for example. Learners could then be set investigative tasks to establish the other key and useful relationships such as tan ($x + \pi$) = tan x and sin ($x + \pi$) = –sin x. Learners could work in groups and, using appropriate software of graphical calculators, investigate what other relationships they can find between the three major trigonometric ratios of $\pi - x$, $\pi + x$, and $2\pi - x$ the three ratios sin$x$, cos$x$ and tan$x$. The results could then be presented to the class as a whole and the findings pooled and written up. (These relationships could also be derived geometrically by transforming the graphs of the respective trigonometric functions if it was felt that learners would respond better to that approach.)	
7.4 **CCSS: F-TF4**	Understand the symmetry (odd and even) and periodicity of trigonometric functions.	The odd and even nature of trig functions can be covered with unit 3.8 or it can be recalled here as useful revision in the context of trig functions. The periodicity of trigonometric functions is most sensibly considered in conjunction with the angles of any magnitude in 7.2. The *GeoGebraTube* activity is ideal to consider the periodicity of trig functions and can also be used to introduce syllabus ref 7.5 and explore 7.6.	 www.geogebratube.org/student/m1395

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
7.5 **CCSS: F-IF4 F-TF5**	Understand amplitude and periodicity and the relationship between graphs of, e.g., sin x and sin $2x$.	Amplitude and periodicity relationships between the graphs of transformed trigonometric functions should be an extension of the transformation of graphs generally, as covered in 0444 extended. Learners should already be able to recognize graphs of $f(x) = a\sin(bx)$; $a\cos(bx)$; tan$x$. Build on this knowledge, and using suitable software, such as *GeoGebra* or graphical calculators, encourage learners to investigate relationships by graphing related graphs and considering those relationships. This would provide ample opportunity for learners to practice working in radian measure also, to increase their familiarity with the concept.	Advanced trigonometry: www.haesemathematics.com.au/samples/ibmyp5plus-2_18.pdf
7.6 **CCSS: F-IF7 F-TF5**	Draw and use the graphs of $y = a \sin bx + c$ $y = a \cos bx + c$ $y = a \tan bx + c$ where a and b are positive integers and c is an integer.	The learning resources mentioned here provide ample opportunity for the transformed graphs of the major trigonometric functions to be considered and relationships explored. Learners should now be drawing together of all the threads they have been working on in relation to transformed trigonometric graphs and reinforcement and practice should aid future recall.	Advanced trigonometry: www.haesemathematics.com.au/samples/ibmyp5plus-2_18.pdf www.geogebratube.org/student/m1395 Specimen Paper 2 Question 11
7.7 **CCSS: F-TF5**	Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.	Learners now have the opportunity to apply the techniques they have honed to practical contexts. They will be expected to both model a situation using a trigonometric function as well as interpret a given function in a particular context. Learners should already have considered and understood amplitude and now need to consider the frequency and midline, although this could also have been considered syllabus ref 7.6. As a starting point ask: • How do modifications in the rule for the cosine function affect the graph? • What particular modifications of the cosine function will produce a function that models the Ferris wheel situation?	Advanced trigonometry: www.haesemathematics.com.au/samples/ibmyp5plus-2_18.pdf

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		The *YouTube* video clip is a nice talk through a Ferris Wheel problem and could be used as an example to work through once learners have considered the essence of what they are trying to do. The *MARS* and *GeoGebraTube* activites are complementary and again work on Ferris wheels. To introduce some variety, the *nctm* activity is concerned with tidal motions and is another detailed investigative task that learners could carry out as a group exercise.	Ferris wheel trig problem: www.youtube.com/watch?v=o7Ho1bMWhG8 MARS – Ferris wheel: http://map.mathshell.org/materials/lessons.php?taskid=427&subpage=concept www.geogebratube.org/student/m14662 nctm – Tidal waves: www.nctm.org/uploadedFiles/Journals_and_Books/Books/FHSM/RSM-Task/Tidal_Waves.pdf
7.8 **CCSS:** **F-TF7** **F-TF8**	Prove and use the relationships $\frac{\sin\theta}{\cos\theta}=\tan\theta, \sin^2\theta+\cos^2\theta=1$	**General guidance** Building on learners understanding of the trigonometric ratios as $\sin x=\frac{\text{opp}}{\text{hyp}}$ $\cos x=\frac{\text{adj}}{\text{hyp}}$ and $\tan x=\frac{\text{opp}}{\text{adj}}$, it is not too much of a stretch for learners to establish $\frac{\sin x}{\cos x}=\frac{\text{opp}}{\text{hyp}}\div\frac{\text{adj}}{\text{hyp}}=\frac{\text{opp}}{\text{hyp}}\times\frac{\text{hyp}}{\text{adj}}=\frac{\text{opp}}{\text{adj}}=\tan x$ Applying the Pythagorean Theorem to an acute, right-angled triangle, the derivation of $\sin^2 x + \cos^2 x = 1$ again should not be too much of a stretch for learners. $b^2=a^2+c^2 \quad (\div b^2)$ $1=\frac{a^2}{b^2}+\frac{c^2}{b^2}$ therefore $1=\left(\frac{a}{b}\right)^2+\left(\frac{c}{b}\right)^2$	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		However, $\frac{a}{b} = \cos\theta$ and $\frac{c}{b} = \sin\theta$ and the result follows. Combining this with the work on angles of any magnitude, the result could then be generalized for any angle.  	Specimen Paper 1 Question 11
	and solve simple trigonometric equations involving the three trigonometric functions and the above relationships (not including general solution of trigonometric equations).	**Notes and exemplars** Includes the use of inverse trigonometric functions. May use inverse functions to solve trigonometric equations that arise in modelling contexts. **General guidance** The notes and examples reference the fact that learners will need to use this particular skill not just in theory but as part of the solution to practical problems. This skill is one, therefore, that underpins others and it is important that learners arrive at a good understanding of the techniques required here. Learners should be reminded that in the exams they are expected to give solutions in degrees to 1 decimal place and angles in radians to 3 significant figures. Start this section by simple practice along the lines of expressing $3\cos^2 x - \sin^2 x$ as a single trig ratio. Once this skill has been sufficiently established look at simple equations such as: • solve the equation cos $2u$ = 0.7 for values of u between 0 and 2π, giving your answers in radians correct to 3 significant figures. Once this has been mastered, learners could move on to solving equations such as	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		• Solve the equation $2\cos^2\theta = 1 - \sin\theta$ for values of θ between 0° and 360°. Ask learners to consider which of the trig ratios they can replace and which they should keep. Once they have recognized the sensible first step, the rest of the method should follow. • Solve the equation $\tan\theta = 3\sin\theta$ for values of θ between 0° and 360°. A common error here is to divide by $\sin\theta$ rather than to factor it out. Careful attention should be given to this. It may prove beneficial to graph each side of the equation using appropriate software and consider the points of intersection prior to considering the algebra of the solution.	
7.9 **CCSS: F-TF9**	Prove and use the expansions of sin ($A \pm B$), cos ($A \pm B$), and tan ($A \pm B$) to solve simple trigonometric problems.	**General guidance** The *RISP* gives a useful introduction to this topic. Using software or graphical calculators, learners are encouraged to select specific trig functions of multiple angles and consider what graph is produced when they are combined in the pattern for addition formulae. Learners could be asked to try to develop a proof for what they have discovered for specific cases. Various proofs of the addition formulae are available – the *Haesemathematics* suggests using two points on the circumference of a unit circle, applying the distance formula and then the cosine rule to establish the result for cos(A – B). The other expansions are derived from this. The *TES* resource is a simple PowerPoint presentation, which offers another approach based upon the area of a triangle using ½ $ab\sin C$.	The compound angle formulae: www.tes.co.uk/ResourceDetail.aspx?storyCode=6056103 Advanced trigonometry: www.haesemathematics.com.au/samples/ibmyp5plus-2_18.pdf Specimen Paper 2 Question 8 Sin(A+B) identity presentation: www.tes.co.uk/ResourceDetail.aspx?storyCode=6030084

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
7.10 **CCSS: F-TF9**	Prove simple trigonometric identities.	**General guidance** The identities that learners will be required to prove will be restricted to those involving the major trig ratios and the relationships already proven in this unit. Learners should bear in mind that when proving a relationship such as $\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x} \equiv \tan x$ they should work from the left to the right and not the left to the right or from each end to the middle, since they must not assume that the steps are reversible. They should consider what they have to show – i.e. the angle has become "*x*" rather than "2*x*" and they should consider what tools they have at their disposal to make some progress in the proof. Here the only really sensible approach is to apply the addition rules to the left and simplify the answer. The need to produce clear and simple proofs should be stressed and a double column approach with statements and reasons as advocated in Unit 4 is advisable. Common errors abound in simplification of expressions such as that on the left and learners must be careful to apply the laws of arithmetic here.	Specimen papers are available at http://teachers.cie.org.uk

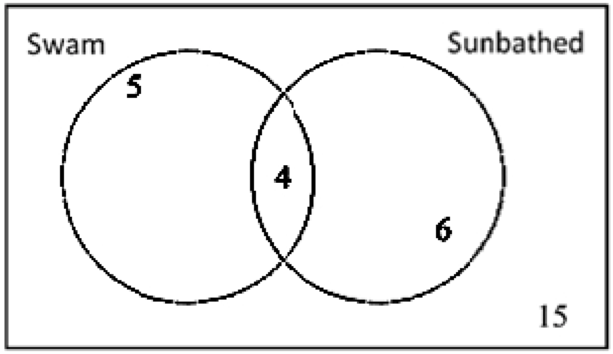
Unit 8: Probability

Recommended prior knowledge
Learners should have covered all elements relating to Probability within 0444 extended syllabus. This syllabus particularly utilizes the elements on combining probabilities. Learners have already covered the notion of independence of events in 0444 extended syllabus. This has entailed them testing
$P(A \text{ and } B) = P(A) \times P(B)$. This needs to be utilized in this unit. They have also covered the notion of mutually exclusive events and considered the implications of $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. Again this will need to be utilized in this unit.

Context
This unit should be taken prior to Unit 9. Apart from this constraint, it can slot into the course at any point. It would be a natural progression to study Units 8 and 9 consecutively.

Outline
The emphasis at this level is on using probability and later statistics to make decisions and simple inferences. Key concepts such as conditional probability and independence of events are covered in this unit and the application of the theory will be the key to achieving success in this unit. Learners will consider the expected value and variance of a probability distribution and analyze decisions and strategies based on their results.

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
8.1 **CCSS:** **S-CP3** **S-CP5** **S-CP6**	Understand and use the conditional probability of A given B as $\frac{P(A \text{ and } B)}{P(B)}$ or as the fraction of B's outcomes that also belong to A; interpret independence of A and B in relation to conditional probabilities and the product of probabilities.	**Notes and exemplar** e.g. compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer. **General guidance** Learners will need to • understand conditional probability and how it applies to real-life events • define and calculate conditional probabilities • understand that events A and B are independent if and only if they satisfy or satisfy $P(A\|B) = P(A)$ or $P(B\|A) = P(B)$ • apply the definition of independence to a variety of chance events	Specimen Paper 1 Question 8

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Examples such as mentioned in the notes and exemplar can be considered using Venn diagrams or two way tables. The fact sheet is a good source of statistics for the Notes and Exemplar example, should it wish to be pursued. Consider other practical problems too, such as: A group of 30 learners went to the beach for the day. 10 sunbathed, 9 went swimming and 15 neither sunbathed nor swam. (i) A learner is selected at random from those who sunbathed. Find the probability that he did not swim. (ii) A learner is selected at random from those who went swimming. Find the probability that she sunbathed.  Learners may well be familiar with Venn Diagrams. If not, the treatment here is informal, but a useful visual aid to learners' understanding and hence mastery of the basic concept of conditional probability. Ask them how they can tell that the groups "swam" and "sunbathed" were not mutually exclusive. Can they determine whether they are independent using the Venn diagram.	Tobacco-related cancers fact sheet: www.cancer.org/Cancer/CancerCauses/TobaccoCancer/tobacco-related-cancer-fact-sheet

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Use all of this to link in A and B are independent if and only if they satisfy or satisfy $P(A\|B) = P(A)$ or $P(B\|A) = P(B)$. This then also ties in neatly with the next syllabus reference. Learners should now be able to conclude that the following four statements are equivalent • A and B are independent events • $P(A \text{ and } B) = P(A) \times P(B)$ • $P(A\|B) = P(A)$ • $P(B\|A) = P(B)$	
8.2 **CCSS: S-CP8**	Apply $P(A \text{ and } B) = P(A)P(B\|A) = P(B)P(A\|B)$ in simple situations, and interpret the answer in context.	**General guidance** Learners should be able to use the general Multiplication Principal, $P(A \text{ and } B) = P(A)P(B\|A) = P(B)P(A\|B)$, to decide if two events are independent and to calculate conditional probabilities. The *Making Statistics Vital* resources, MSV 27 and 32 are excellent investigative activities reinforcing the independence of events.	*Making statistics vital* - Statistics: www.making-statistics-vital.co.uk/
8.3 **CCSS: S-CP9**	Use permutations and combinations to compute probabilities of compound events and solve problems.	**General guidance** Learners need to be able to identify situations as appropriate for use of a permutation or combination to calculate probabilities. This is an extension of the counting techniques already mastered. Now the number of permutations or combinations of a subgroup from a finite population is considered. The factorial function, n! may need to be defined first. The *TES* resource starts this topic by considering the factorial function, so this may prove useful as a starting point. *Better Explained* tackles the topic from quite an intuitive viewpoint **Permutations are for lists (order matters) and combinations are for groups (order doesn't matter)**. The definitions of perms and combs are then backed up with a simple example allocating medals and so on, which learners should follow quite intuitively. This type of approach is useful as learners will find it more memorable.	*TES* Statistics: www.tes.co.uk/teaching-resource/Statistics-1-Arrangements-6033012/ Easy permutations and combinations: http://betterexplained.com/articles/easy-permutations-and-combinations/

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Teaching activities The *TES* resource is a PowerPoint introducing the idea of combinations using the UK National Lottery and then broadening the theme. Learners are encouraged to work as teams and devise their own lottery. A useful hands-on introduction to the topic.	Permutation and combination with the lottery: www.tes.co.uk/teaching-resource/Permutation-and-Combination-with-the-Lottery-6126510/
8.4 **CCSS:** **S-MD1** **S-MD2** **S-MD3** **S-MD5**	Define a random variable X by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions; and calculate $E(X)$ and $Var(X)$.	**General guidance** Notation • the random variable generally is referred to with e.g. uppercase X • specific values of the distribution are denoted with e.g. lowercase x • $P(X = x)$ or P_x refers to the probability that the random variable X is equal to the specific value x. • $E(X)$ is the mean of X calculated from a probability distribution Learners should be able to, given a probability situation (theoretical or empirical): • define a random variable as a quantity that can take any value determined by the outcome of a random event • understand what the properties of a random variable are – they are numerical; they can be discrete of continuous • create a table and graph of the probability distribution of the random variable • understand that the probabilities in a probability distribution should total 1 • use the table to evaluate the expected value of the random variable by multiplying all possible values by their probabilities and summing the results • use the table to evaluate the variance of the random variable, by first finding the expected value of X^2 by multiplying the square of all possible values by their probabilities and summing the results and then applying $Var(X) = E(X^2) – [E(X)]^2$ Learners will already be familiar with frequency distributions. Probability distributions should be a natural progression from frequency distributions. It is straightforward to link the idea of finding the mean of a	Specimen Paper 1 Question 13

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		frequency distribution by multiplying to obtain subtotals, summing to obtain a grand total and then dividing by the total frequency to finding the expected value of a probability distribution by dividing each frequency first to obtain relative frequency, then multiplying to obtain subtotals and then summing for the final answer. Making this connection to known concepts, and considering probability as the limiting value or long term value of relative frequency, should help learners to come to a better fundamental grasp of what they are trying to achieve. For example, let X be the number of heads obtained when a coin is tossed twice. Ask learners what possible values X could have. Set up a table for the probability distribution e.g. x: 0, 1, 2 $P(X = x)$: ¼, ½, ¼ The expected value can be considered then as the long term average. Calculated E(X) from the table and ask the question – will the expected value always be one of the specific values the random variable can take?	
		The (population) variance should be a new concept. Its use as a measure of dispersion will need to be justified. Starting with a simple list such as 0 9 9 9 10 10 10 10 11 11 21 mean = median = mode = 10, LQ = 9, UQ = 11, IQR = 2 and range = 21 Consider another list 0 0 9 9 10 10 10 10 11 20 21 mean = median = mode = 10, LQ = 9, UQ = 11, IQR = 2 and range = 21 The second data set has more extreme values than the first and the measures of location and dispersion used do not reflect this.	

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<tr><th>Syllabus ref and CCSS</th><th>Learning objectives</th><th>Suggested teaching activities</th><th>Learning resources</th></tr>
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Considering the difference between each data value and the mean we have, for each set

First	–10	–1	–1	–1	0	0	0	0	1	1	11
Second	–10	–10	–1	–1	0	0	0	0	1	10	11

We are only interested in the difference, not the sign of the difference. One way to achieve this is to square the differences. Any large differences will be magnified by doing this.

First	100	1	1	1	0	0	0	0	1	1	121
Second	100	100	1	1	0	0	0	0	1	100	121

Summing and averaging (to take account of the number of data values in each list should it be different)

Variance First = 226 ÷ 11 = 20.5454…
Variance second = 424 ÷ 11 =38.54…..

This will need to be generalised (the explanation is simplified if learners understand the Σ notation to mean “sum”) and the idea of working form a frequency distribution introduced:

$$\text{Variance} = \frac{\sum (x = \text{mean})^2}{n} = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = \ldots = \frac{\sum x^2 f}{n} - \left(\frac{\sum xf}{n}\right)^2$$

This form of the variance can now be extended to probability distributions with

$$\mathrm{E}(x^2) = \frac{\sum x^2 f}{n} = \sum x^2 \mathrm{P}_x \text{ and } [\mathrm{E}(x)]^2 = \left(\frac{\sum xf}{n}\right)^2 = \left(\sum x\mathrm{P}_x\right)^2 \text{ and so}$$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

Note: It may be useful at this stage, as a precursor to later work, to mention that the variance is the average of the squared distances from the mean. The standard deviation is the square root of the variance and therefore is a measure of the spread of the data that has the same units

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Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		as the data.	
		Now find the variance of the number of heads obtained when a coin is tossed twice. Finding E(X^2) as above and using that, find the Var(X). **Teaching activities** It will be essential that learners have plenty of practice in setting up probability distributions and using them to find the expected value and the variance. The *Making Statistics Vital* resource, MSV 31, provides opportunity for learners to investigate generally E(X) and Var(X).	Making statistics vital: www.making-statistics-vital.co.uk/
8.5 **CCSS:** **S-MD4** **S-MD5**	Investigate the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. Find the expected payoff values for a game of chance. Evaluate and compare strategies on the basis of expected values.	**Notes and exemplars** e.g. • Find the expected winnings from a state lottery ticket. • Compare a high-deductible versus a low deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident. **General guidance** The simple probability distribution for the number of heads obtained when two fair coins are tossed set up previously can be utilized here as a simple basis for a game of chance. For example, you pay \$2 to play the game. The two coins are tossed and if both land Heads up you win a \$5 prize, otherwise you lose. Suppose you can play this game as many times as you like. Would you play over and over again? Does either side have an advantage? If so, what is it? Let the random variable X be the amount won. <table><tr><td>*Heads*</td><td>0</td><td>1</td><td>2</td></tr></table>	

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<tr><th>Syllabus ref and CCSS</th><th>Learning objectives</th><th>Suggested teaching activities</th><th>Learning resources</th></tr>
<tr><td></td><td></td><td>
<table>
<tr><td>X</td><td>-$2</td><td>-$2</td><td>$3</td></tr>
<tr><td>P(X = x)</td><td>¼</td><td>½</td><td>¼</td></tr>
</table>
Now E(X) = $\left(-2\times\frac{1}{4}\right)+\left(-2\times\frac{1}{2}\right)+\left(3\times\frac{1}{4}\right)=-0.5\ -1+0.75=-0.75$
If you play this game over and over again, you will lose an average of 75¢ per game. The other side has the advantage in this game since they win an average of 75¢ each time the game is played.
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<tr><td>8.6

CCSS: S-MD7</td><td>Analyze decisions and strategies using probability concepts.</td><td>
Notes and exemplar
e.g. product testing, medical testing, pulling a hockey goalie at the end of a game.

General guidance
This lends itself naturally to investigation within the classroom. Lots of activities can be carried out or simulated using software.

Teaching activities
For example, weigh the contents of many small candy bars. Find the numerical summaries of the weights of the bars and plot the results. How do the weights compare to the manufacturer's stated weight? How likely is getting a candy bar at or above the manufacturer's stated weight?

Extend to more complex probability models. Include situations such as those involving quality control, or diagnostic medical tests that produce both false positive and false negative results.

This *GeoGebra* worksheet can be used to explore the following problem, which is a classic application of Bayes' theorem: If a person tests positive for a disease, what is the probability that he or she is actually infected?

The *mathshell* activities are intended assess how well learners are able to understand conditional probability; represent events as a subset of a sample space using tables and tree diagrams; communicate their
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www.geogebratube.org/student/m4054

Medical testing:
http://map.mathshell.org/materials/lessons.php?taskid=438&subpag
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Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		reasoning clearly. The *Khan Academy* video presents a life insurance scenario.	e=problem Modeling conditional probabilities: http://map.mathshell.org/materials/lessons.php?taskid=409&subpage=problem Law of large numbers: www.khanacademy.org/math/probability/v/term-life-insurance-and-death-probability Specimen papers are available at http://teachers.cie.org.uk

Unit 9: Statistics

Recommended prior knowledge
Learners should have studied all of the units relating to probability and statistics in 0444 extended syllabus.

Context
This unit should be studied after Unit 8 and also Units 3 and 6, as elements of fitting a linear model will possibly require the skills in the logarithmic section as well as in the coordinate geometry unit. Subsection C of this unit could be taught alongside Unit 8 (8.1).

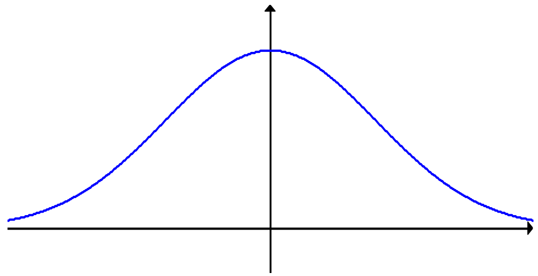
Outline
Learners have already met different ways of collecting data and using graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. Consideration is given to sampling techniques in subsection A and the effect they have on the results produced. The Normal distribution is introduced in subsection B and the mechanics of it mastered and the importance of it to real-life situations emphasized. In subsections C and D, the focus is that two categorical or two quantitative variables are being measured on the same subject, with analysis of the results.

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
Subsection A (Sampling)			
		The *Amstat Level B and C* pdfs are full of good ideas that can be utilized throughout this unit. They are highly recommended reading prior to undertaking subsection A.	Amstat: www.amstat.org/education/gaise/GAISEPreK12_LevelB.pdf www.amstat.org/education/gaise/GAISEPreK12_LevelC.pdf
9.1 **CCSS:** **S-IC1** **S-IC3**	Understand the concept of sampling and recognize the purposes of and differences among sample surveys, experiments, and observational	**General guidance** Discuss with learners: When carrying out a survey you will generally be dealing with a finite population e.g. the electorate of the US. In an ideal world, it would be	Specimen Paper 1 Question 2

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
	studies; explain how randomization relates to each.	possible to carry out a survey on the whole population (c.f. Census), but this carries with it a number of problems. What are they do you think? Mainly • the organisation required and the high cost • the processing of the results would be lengthy and therefore the information would be out of date. (List all problems learners come up with.) Learners should then appreciate why It is therefore better to take a smaller sample of the population and estimate the required results from the sample. The problem arises as to how well the sample actually represents the population. Learners should have the opportunity to: • Identify situations as ones where sample survey, experiment, or observational study would be appropriate and discuss the appropriateness of each one's use in contexts with limiting factors. • Design or evaluate sample surveys, experiments and observational studies with randomization. • Discuss the importance of randomization in these processes. Random sampling is a way to remove bias in sample selection, and tends to produce representative samples. If a sample is random there should be no bias whatsoever in the method of sample selection. How could this be achieved? E.g. random number generators, picking lots from a 'hat', etc. Other, non-random methods of sampling should then be considered and discussed such as Stratified sampling, Systematic sampling, Quota sampling, Cluster sampling.	
9.2 **CCSS:** **S-IC4** **S-IC5**	Use data from a sample survey to estimate a population mean or proportion; use data to compare two variables.	When we sample from a finite population, we usually wish to estimate: i) the average value of some characteristic (such as the mean) ii) the proportion lying in some category Learners should understand values describing a population and a	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		sample will not be exactly the same. Discuss with them what affects the results for samples. What makes a sample useful? • Unbiased selection • The correct population being sampled • A sensible sample size being taken The concepts in Unit 8 (8.4) will be useful here with probability seen as long term relative frequency and with the expected value being seen as the long term mean. To aid this explanation it would be useful to carry out an activity with the class where they each take a sample from a population and compare results. The 80 circles activity on pages 17–19 of the *Amstat Level B* pdf is a simple way to help learners grasp the concept of unbiased selection. The activity could be extended to consider the effects of sample size, with learners being grouped together into different sized groups and the results compared. From this learners should appreciate that a representative sample is one in which the relevant characteristics of the sample members are generally the same as those of the population. Improper or biased sample selection tends to systematically favor certain outcomes, and can produce misleading results and erroneous conclusions.	Amstat: www.amstat.org/education/gaise/GAISEPreK12_LevelB.pdf
9.3 **CCSS: S-ID3**	Interpret differences in shape, center, and spread in the context of data sets, accounting for possible effects of outliers.	**General guidance** This builds on the work learners have already undertaken in this area in 0444 extended syllabus (10.7) where learners learn that quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. Now they need to consider that the shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as variance, standard deviation or interquartile range). Learners should know that outliers are extreme values, sometimes the result of faulty data, and they are common in skewed distributions.	Interpret differences in shape, center…..: https://ccgps.org/S-ID_625Z.html

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		Recalling the definition of the Variance of a data set as average of the squared distances from the mean, define the Standard Deviation as the square root of the variance and therefore is a measure of the spread of the data that has the same units as the data. An outlier could then be considered as a point which is without two standard deviations of the mean.	
		In this *Amstat* activity, learners collect data from an experiment designed to investigate whether or not the ability to name the color of printed ink is challenged when the word written with the ink is the name of a color. Learners are asked to design the experiment to investigate this. Numeric summaries (mean and five-number summary) and comparative boxplots are used to summarize the collected data. Conclusions are drawn about the difficulty of naming a color when it is displayed in this manner.	Amstat: www.amstat.org/education/stew/pdfs/ColorsChallenge.docx
Subsection B (Introducing the Normal Distribution)			
9.4 **CCSS: S-ID4**	Use standardized values and normal tables for normally distributed continuous data in determining probabilities as areas under the normal curve.	**General guidance** The normal distribution is the most important of all distributions since it describes the situation in which extreme values (very large or very small) are rare but values in the middle are rather commonplace. As this describes many real life scenarios it is “normal”! Its properties are: • it relates to continuous data (e.g. heights, masses, lengths, times..) • it is symmetrical about its mean • it is infinite in both directions • 95% of values lie with approximately two standard deviations of the mean • 99% of values lie within approximately three standard deviations of the mean • it has a bell shaped curve.	Specimen Paper 2 Question 9 Amstat – What does the normal distribution sound like?: www.amstat.org/education/stew/pdfs/WhatDoestheNormalDistributionSoundLike.pdf

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		 Compare histograms with frequency density to normal curves with probability density. This should help learners understand why the area under a normal curve is a measure of the probability between two points of the distribution. Introduce the notation: If X is a normally distributed random variable with mean μ and variance σ^2 then we write $X \sim \mathrm{N}(\mu, \sigma^2)$. Since there are infinitely many different normal distributions, and to calculate the probabilities of each individually would be tedious, and also make comparisons difficult, the Standard Normal is introduced. Justify this by explaining that, for any distribution, changes in μ and σ can be thought of as changes of location and scale. The standard normal has mean 0 and standard deviation (and therefore variance) of 1. Notation for standard normal: $Z \sim \mathrm{N}(0, 1)$ To scale any normal distribution to the standard normal learners should learn $Z = \dfrac{X - \mu}{\sigma}$ The *GeoGebraTube* activity allows learners to make the visual connections between the standard normal and various other normal distributions and also has a useful tool for evaluating probabilities. Once learners have been introduced to the standard normal and how the standard normal probability tables they will be expected to use for	www.geogebratube.org/material/show/id/11896

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		their examinations work, they should experience as much practice as possible, in the form of simple theoretical tasks to master the skills they will need for assessment as well as investigative work utilizing the normal distribution in practical circumstances. Work can be set as investigations using graphical calculators or other appropriate software. Plenty of pen and paper practice should be given so that learners grasp the mechanisms of what they are attempting successfully. Drawing a bell curve in order to use the tables successfully is an essential skill that will be mastered with practice.	
9.5 **CCSS: S-ID4**	Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate.	Learners now use the normal distribution to make specific estimates. Build on learners' understanding of data distributions to help them see how the normal distribution uses area to make estimates of frequencies (which can be expressed as probabilities). Emphasize that only some data are well described by a normal distribution, for example bimodal data would not be well described by a normal fit.	
Subsection C (Two Way Tables)			
9.6 **CCSS: S-ID5**	Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.	**General guidance** Here the focus is that two categorical variables are being measured on the same subject. • Begin with two divisions for each category and represent them in a two-way table (so you will produce a 2 by 2 table – extend the idea up to 3 by 3 perhaps once 2 by 2 has been mastered) • (Extending the number of rows and columns is easily done once learners become comfortable with the 2×2 case.) • The table entries are the joint frequencies (how many of the item under consideration (often people) fit both categories, row and column) • Row totals and column totals are the marginal frequencies • Dividing joint or marginal frequencies by the total number define relative frequencies (and percentages), respectively.	Two-way tables in statistics: http://stattrek.com/statistics/two-way-table.aspx

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		• Conditional relative frequencies are determined by focusing on a specific row or column of the table. They are particularly useful in determining any associations between the two variables. The *NCSU* activity looks closely at data analysis for two way tables. In the first case and then moves on to association and the question of causation which links nicely into subsection D (Scatter Plots).	NCSU – Looking at data – relationships: http://www4.ncsu.edu/~muse/Teaching/ST305/Lectures/Ch02-pt2.pdf
9.7 **CCSS: S-CP4**	Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.	**General guidance** For example, collect data from a random sample of learners in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected learner from your school will favor science given that the learner is in tenth grade. Do the same for other subjects and compare the results. This really draws together the skills in syllabus ref 9.6 and also in unit 8 regarding conditional probability and independence. The emphasis here should be on interpreting results and making comparisons.	
Subsection D (Scatter plots)			
9.8 **CCSS: S-ID6 S-ID7**	Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. • Fit a function to the data; use functions (e.g., linear, quadratic, exponential) fitted to data to solve problems in the context of the data. • Informally assess the fit of a function by plotting and analyzing residuals. • Fit a linear function for a scatter plot that suggests a linear association. • Fit a function to the data; use	**General guidance** Learners will already be familiar with fitting a linear function to data from 0444 extended syllabus. They should already be able to draw a line of best fit, appreciating the need for it to mass through the mean and describe its correlation. Here, the level of difficulty is increased, so that learners are expected to work with non-linear data. They will need to be able to find the equation of a line of best fit if the association is linear or fit a suitable quadratic or exponential curve to non-linear data as appropriate. Learners should able to use the Statistics function on their calculators to find least squares regression lines and compare the equation the calculator gives them for their data to the equation of the line they have drawn. This should give a natural introduction to the discussion of residuals and the informal assessment and analysis of residuals.	Specimen Paper 2 Question 5 www.geogebratube.org/student/m

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
	functions fitted to data to solve problems in the context of the data. • Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.	The *GeoGebraTube* activity then builds on more analysis of residuals by undertaking a plot and considering the patterns made if there are any. The *CCSS Toolbox* activity is also a useful visual presentation to aid learners' learning here. It is a common misconception among learners that a pattern indicates a good fit! For linear models, they should be able to interpret the gradient of their line in the context of the problem set – so for example, one extra hour's homework produces five extra grade points in a test for a positive slope of 5. This topic area may be linked with Unit 3 subsection C (Logarithmic and Exponential functions) as learners may be asked to confirm that an exponential fit is appropriate by converting to straight line form and assessing the resulting linear fit.	14014 Key visualizations: Algebra 1: http://ccsstoolbox.agilemind.com/animations/standards_content_visualizations_algebra_1.html
9.9 **CCSS:** **S-ID8** **S-ID9**	Interpret the correlation coefficient of a linear fit and distinguish between correlation and causation.	**General guidance** Learners will be expected to be able to find the product moment correlation coefficient, *r*, using their calculators. Either having been given the correlation coefficient or having found it, they should be able to determine what it means regarding the data under scrutiny. Learners should appreciate that different samples from the same population will produce different values of the correlation coefficient. The variation between these values reduces as the sample size increases, but for this reason, it is not sensible to be overly-accurate with the value of *r* quoted. They should also appreciate the effect an outlier can have on the value of *r* and that drawing a scatter graph is always a sensible idea to be wary of outliers. It is vital that learners understand that correlation does not imply causation. A third (hidden or confounding) variable may be a factor in the apparent link between the two variables. For example, the number of ice-creams sold at the beach in August and the number of life-guard rescues in August may have a positive correlation. However, it is highly	

Syllabus ref and CCSS	Learning objectives	Suggested teaching activities	Learning resources
		unlikely that the increase in ice-cream sales is the cause of the increase in rescues. A third variable – for example the temperature in August being hotter than other months – is most likely to be behind each of these increases.	
CCSS: S-IC6		The *CPM* pdfs are a super resource for subsection D. Learners will begin by collecting data that is scattered due to natural measurement variability. They model the relationship between the variables using a line of best fit which they draw by eye and interpret the slope and *y*-intercept of the model in the context of the problem. Learners then explore the limitations of the models when extrapolated far from the edges of the actual data. Since variability in data cannot be avoided, learners begin learning to quantify variability. They describe the association in the data verbally at first, then quantify it by creating a consistent linear regression using their calculators. They will graph residuals and calculate upper and lower bounds, correlation coefficients, and *R*-squared, and interpret these quantities in the context of the situation. Learners look at residual plots to assess the fit of a linear model to their data. In the process of quantifying variability, learners collect data from another study: in the anthropologist study, variability is due not only to measurement variability, but also to element variability – the natural variability between humans. Learners consider the physical context of the association and use their calculators to create an appropriate non-linear model that fits the data and the situation. They use residual plots – along with their common sense about the relationship – to assess the fit of various models.	CPM – Modeling two-variable data: www.cpm.org/pdfs/standards/stats/Stats%20Unit%207%20TV.pdf Specimen papers are available at http://teachers.cie.org.uk